

Fluctuations of TASEP and LPP with general initial data

Ivan Corwin* Zhipeng Liu† and Dong Wang‡

December 16, 2014

Abstract

We prove Airy process variational formulas for the one-point probability distribution of (discrete time parallel update) TASEP with general initial data, as well as last passage percolation from a general lattice path to a point. We also consider variants of last passage percolation with inhomogeneous weights and provide variational formulas of a similar nature. This proves one aspect of the conjectural description of the renormalization fixed point of the Kardar-Parisi-Zhang universality class.

1 Introduction

The totally asymmetric simple exclusion process (TASEP) is a prototypical interacting particle system, or (via integration) random growth process. The theory of hydrodynamics describes the law of large number for the evolution of the system's particle density, or height function. In particular, if $h(x; t)$ represents the height function, then $\epsilon h(\epsilon^{-1}x; \epsilon^{-1}t)$ converges (as $\epsilon \rightarrow 0$) as a space-time process to the deterministic solution to a Hamilton Jacobi equation with explicit (model dependent) flux [27]. The solution, of course, depends on the initial data and in particular on the limit (as $\epsilon \rightarrow 0$) of $\epsilon h_0(\epsilon^{-1}x)$. It is possible to consider initial data $h_{0;\epsilon}$ which depends on ϵ so that $\epsilon h_{0;\epsilon}(\epsilon^{-1}x)$ has a limit.

The aim of the present paper is to describe, in a similar spirit, how fluctuations around the law of large number evolves over time. Define

$$h^\epsilon(x; t) = c_1 \epsilon^b h(c_2 \epsilon^{-1}x; c_3 \epsilon^{-z}t) - \bar{h}^\epsilon(x; t).$$

Then it is conjectured in [18] that if we take

$$b = 1/2, \quad \text{and} \quad z = 3/2,$$

*Columbia University, Department of Mathematics, 2990 Broadway, New York, NY 10027, USA, and Clay Mathematics Institute, 10 Memorial Blvd. Suite 902, Providence, RI 02903, USA, and Institute Henri Poincare, 11 Rue Pierre et Marie Curie, 75005 Paris, France, and Massachusetts Institute of Technology, Department of Mathematics, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA.

email: ivan.corwin@gmail.com

†Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA
email: zhipeng@cims.nyu.edu

‡Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Rd, Singapore 119076.
email: matwd@nus.edu.sg

then for c_1, c_2, c_3 model dependent constant (chosen in terms of microscopic dynamics in terms of the KPZ scaling theory [35, 28]) and suitable centering $\bar{h}^\epsilon(x; t)$ (coming from the hydrodynamic theory) the space-time process $h^\epsilon(\cdot; \cdot)$ will have a universal limit $\mathfrak{h}(\cdot; \cdot)$ which is independent of the underlying model. The class of all models which satisfy this is called the Kardar-Parisi-Zhang universality class, and this limiting object is called the fixed point of this universality class.

Much of the description and almost all of the universality of this fixed point remains a matter of conjecture. One of the main conjectures provided in [18] (see also the review [33]) about this fixed point is that its solution can be described via a variational problem (in the spirit of the Lax-Oleinik formula for the inviscid Burgers equation) involving a four-parameter random field called the space-time Airy sheet. A corollary of this conjectural description is that if $h^\epsilon(\cdot, 0)$ converges (as a spatial process) to some function $\mathfrak{h}_0(\cdot)$, then we have the following distributional equality, valid for any single pair of fixed x :

$$\mathbb{P}(\mathfrak{h}(x, 1) \geq -r) = \mathbb{P}\left(\max_{y \in \mathbb{R}} (\mathcal{A}(y) - (x - y)^2 - \mathfrak{h}_0(y)) \leq r\right).$$

Here $\mathcal{A}(\cdot)$ is the Airy process (Section 2.6) and by scaling properties of \mathfrak{h} , this implies a similar conjecture for general t .

The main contribution of the present paper is a proof of this conjectured variational description for the limiting one-point distribution of TASEP. In particular, consider the parameter q discrete time parallel update TASEP height function (Section 2.3) and define

$$h^{\epsilon, \text{TASEP}}(x; t) = \frac{\epsilon^{1/2} h(2\mathbf{c}_0 \epsilon^{-1} x; \mathbf{a}_0^* \epsilon^{-3/2} t) - 2\epsilon^{-1} t}{2\mathbf{d}_0^*}$$

where

$$\mathbf{a}_0^* = \frac{2}{1 - \sqrt{q}}, \quad \mathbf{c}_0 = \frac{(1 + \sqrt{q})^{2/3}}{q^{1/6}}, \quad \mathbf{d}_0^* = \frac{q^{1/6}(1 + \sqrt{q})^{1/3}}{2}.$$

Theorem 2.8 shows that if $h^{\epsilon, \text{TASEP}}(\cdot, 0)$ converges in distribution (as a spatial process) to some function $\mathfrak{h}_0(\cdot)$ then (subject to certain growth hypotheses at infinity)

$$\lim_{\epsilon \rightarrow 0} \mathbb{P}(h^{\epsilon, \text{TASEP}}(x, 1) \geq -r) = \mathbb{P}\left(\max_{y \in \mathbb{R}} (\mathcal{A}(y) - (x - y)^2 - \mathfrak{h}_0(y)) \leq r\right).$$

In order to prove this we first relate the TASEP height function one-point distribution to a discrete variational problem called point-to-curve geometric last passage percolation (LPP). The particular curve in question encodes the height function initial data. LPP has been studied previously, and, in particular, Johansson [26] proved that the Airy process $\mathcal{A}(\cdot)$ minus a parabola describes the spatial fluctuations of point-to-point LPP as one point varies along an anti-diagonal line. In order to extend Johansson's result away from the anti-diagonal line and onto a general curve we prove a uniform slow decorrelation result, which shows that up to deterministic shift (related to the given curve) the fluctuations along the line and along a general curve agree. The final step in proving our result is to conclude that the resulting variational problem stays localized as ϵ goes to zero and this is achieved via a combination of large/moderate deviation bounds on TASEP and a utilization of some regularity estimates coming from the Gibbs property of the associated multi-layer PNG line ensemble (Section 6).

In a similar manner we prove variational one-point distribution formulas for point to general curve LPP as well as LPP in which some of the weights have been perturbed. As a corollary of the TASEP and LPP results we provide variational formulas for a number of known one-point distributions, such as arise in TASEP with combinations of wedge, flat and stationary initial data.

Organization of the paper

Section 2 introduces the models (LPP and TASEP) as well as the main results (Theorems 2.6, 2.8, 2.11 and 2.13) about them. The proofs of these theorems are applications of Theorem 2.18 on the uniform slow decorrelation and Theorem 2.19 on the Gibbs property, and they are given in Section 3. Proofs of corollaries 2.9 and 2.14 are given in Section 4. The technical results, Theorems 2.18 and 2.19, are proved in Sections 5 and 6 respectively. Finally the appendix gives the proof of Lemma 2.2.

Acknowledgements

The authors thank to Jinho Baik, Jeremy Quastel and Daniel Remenik for fruitful discussions. Ivan Corwin was partially supported by the NSF through DMS-1208998 as well as by Microsoft Research and MIT through the Schramm Memorial Fellowship, by the Clay Mathematics Institute through the Clay Research Fellowship, by the Institute Henri Poincare through the Poincare Chair, and by the Packard Foundation through a Packard Foundation Fellowship. Zhipeng Liu gratefully acknowledges the support from the department of mathematics, University of Michigan. Dong Wang was partially supported by the startup grant R-146-000-164-133.

2 Models and main results

2.1 Point-to-curve LPP

Associate to each site $(i, j) \in \mathbb{Z}^2$ an independent geometrically distributed random variable $w(i, j)$ with parameter $1 - q$, such that

$$\mathbb{P}(w(i, j) = k) = (1 - q)q^k, \quad k = 0, 1, 2, \dots \quad (1)$$

The point-to-point last passage time between two lattice points (x, y) and (x', y') is denoted by $G_{(x', y')}(x, y)$ and defined by

$$G_{(x', y')}(x, y) := \begin{cases} \max_{\pi} \left\{ \sum_{(i, j) \in \pi} w(i, j) \mid \pi \in (x, y) \nearrow (x', y') \right\} & \text{if } x \leq x' \text{ and } y \leq y', \\ -\infty & \text{otherwise,} \end{cases} \quad (2)$$

where π stands for an up-right path such that $\pi = (\pi_0 = (x, y), \pi_1, \pi_2, \dots, \pi_{x'+y'-x-y} = (x', y'))$ and $\pi_{k+1} - \pi_k \in \{(1, 0), (0, 1)\}$. More generally, if (x', y') is a lattice point, and (x, y) is on a line segment between two neighboring lattice points, then define

$$G_{(x', y')}(x, y) := \text{the linear interpolation between } \begin{cases} G_{(x', y')}(x, [y]) \text{ and } G_{(x', y')}(x, [y] + 1) & \text{if } x \in \mathbb{Z}, \\ G_{(x', y')}([x], y) \text{ and } G_{(x', y')}([x] + 1, y) & \text{if } y \in \mathbb{Z}. \end{cases} \quad (3)$$

If (x, y) and (x', y') are lattice points, we define the short-handed notations for the reversed last passage time as

$$\check{G}_{(x', y')}(x, y) = G_{(x, y)}(x', y') \quad \text{and} \quad \check{G}(x, y) := \check{G}_{(0, 0)}(x, y) = G_{(x, y)}(0, 0). \quad (4)$$

We also define $\check{G}_{(x',y')}(x, y)$ by linear interpolation if (x, y) is on a line segment between two neighboring lattice points, analogous to (3). We will consider a more general point-to-curve last passage time, denoted by $G_{(x',y')}(L)$ in this paper. Let (x', y') be a lattice point and L be a lattice path in \mathbb{R}^2 with $L = L(s) = \{(x(s), y(s)) \mid s \in I\}$, for some interval $I \subset \mathbb{R}$. Here a lattice path means a directed path composed by line segments each of which connects two neighboring lattice points. Define

$$G_{(x',y')}(L) = \sup_{s \in I} \{G_{(x',y')}(x(s), y(s))\}. \quad (5)$$

Although s is a continuous parameter, it suffices to take the supremum among a discrete set of point-to-point last passage times.

As preliminaries for our work, let us recall some important results about the asymptotic behavior of the point-to-point and point-to-curve last passage time. Focusing first on point-to-point last passage percolation, we state the law of large numbers, large/moderate deviations and the fluctuation limit theorems in the following proposition. Note that due to the symmetry of the lattice, we state our results in terms of $\check{G}(x, y)$.

Proposition 2.1. *Let γ be in a compact subset U of $(0, \infty)$. Then*

(a) (Johansson [25])

$$\lim_{N \rightarrow \infty} \frac{1}{N} \check{G}(\gamma N, N) = a_0(\gamma), \quad \text{almost surely, where } a_0(\gamma) = \frac{(\gamma + 1)q + 2\sqrt{\gamma q}}{1 - q}. \quad (6)$$

(b) (Baik-Deift-McLaughlin-Miller-Zhou [3]) *There exist a (large) constant $M > 0$ and a (small) constant $\delta > 0$ such that for large N , uniformly for all $M \leq x \leq \delta N^{1/3}$, there exists $c > 0$ such that*

$$\mathbb{P} \left(\check{G}(\gamma N, N) \leq a_0(\gamma)N - xN^{1/3} \right) \leq e^{-cx^3}. \quad (7)$$

(c) (Johansson [25])

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{\check{G}(\gamma N, N) - a_0(\gamma)N}{b_0(\gamma)N^{1/3}} \leq x \right) = F_{\text{GUE}}(x), \quad (8)$$

where

$$b_0(\gamma) = \frac{q^{1/6}\gamma^{-1/6}}{1 - q} (\sqrt{q} + \sqrt{\gamma})^{2/3} (1 + \sqrt{\gamma q})^{2/3}, \quad (9)$$

and $F_{\text{GUE}}(x)$ is the Tracy-Widom distribution for the limiting fluctuation of the largest eigenvalue in the Gaussian unitary ensemble (GUE), see Section 2.6.

In this paper, we need a counterpart of (7), which is stated below, and proved in Appendix A.

Lemma 2.2. *Let γ be in a compact subset U of $(0, \infty)$. There exist a (large) constant $M > 0$ and a (small) constant $\delta > 0$ such that for large N , uniformly for all $M \leq x \leq \delta N^{1/3}$, there exists $c > 0$ such that*

$$\mathbb{P} \left(\check{G}(\gamma N, N) \geq a_0(\gamma)N + xN^{1/3} \right) < e^{-cx}. \quad (10)$$

We denote in this paper

$$\mathbf{a}_0 = a_0(1) = \frac{2\sqrt{q}}{1 - \sqrt{q}}. \quad (11)$$

Define the limit shape curve (see Figure 1)

$$\begin{aligned} \check{\mathcal{L}} &:= \left\{ (x, y) \in (0, \infty) \times (0, \infty) \mid ya_0 \left(\frac{x}{y} \right) = \mathbf{a}_0 \right\} \\ &= \left\{ (r(\theta) \cos \theta, r(\theta) \sin \theta) \mid \theta \in (0, \frac{\pi}{2}) \text{ and } r(\theta) = \frac{2(1 + \sqrt{q})}{(\cos \theta + \sin \theta)\sqrt{q} + 2\sqrt{\cos \theta \sin \theta}} \right\}, \end{aligned} \quad (12)$$

and

$$\mathcal{L} := \{(x, y) \mid (1 - y, 1 - x) \in \check{\mathcal{L}}\}. \quad (13)$$

Note that

$$\frac{\mathbf{a}_0}{\lim_{x \rightarrow 0} a_0(x)} = 2 + 2q^{-1/2} \quad \text{and} \quad 1 - \frac{\mathbf{a}_0}{\lim_{x \rightarrow 0} a_0(x)} = -1 - 2q^{-1/2}, \quad (14)$$

so $\check{\mathcal{L}}$ (\mathcal{L} resp.) is between $(2 + 2q^{-1/2}, 0)$ and $(0, 2 + 2q^{-1/2})$ ($(-1 - 2q^{-1/2}, 1)$ and $(1, -1 - 2q^{-1/2})$ resp.). Then by Proposition 2.1(a) we have that if $(x, y) \in \check{\mathcal{L}}$, then $\check{G}([xN], [yN]) = \mathbf{a}_0 N + o(N)$, or equivalently, if $(x, y) \in \mathcal{L}$, then $G_{(N,N)}([xN], [yN]) = \mathbf{a}_0 N + o(N)$.

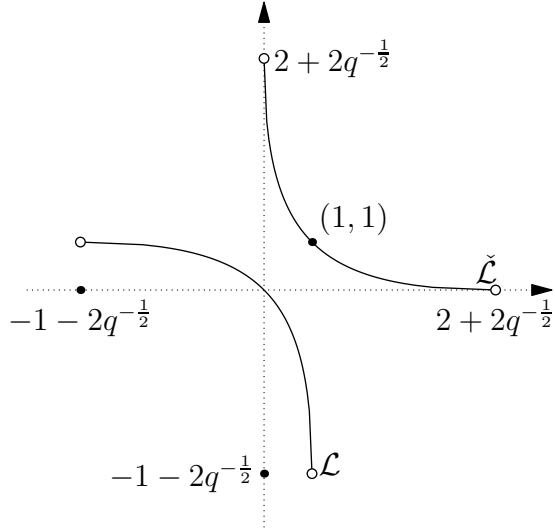


Figure 1: The shapes of \mathcal{L} and $\check{\mathcal{L}}$.

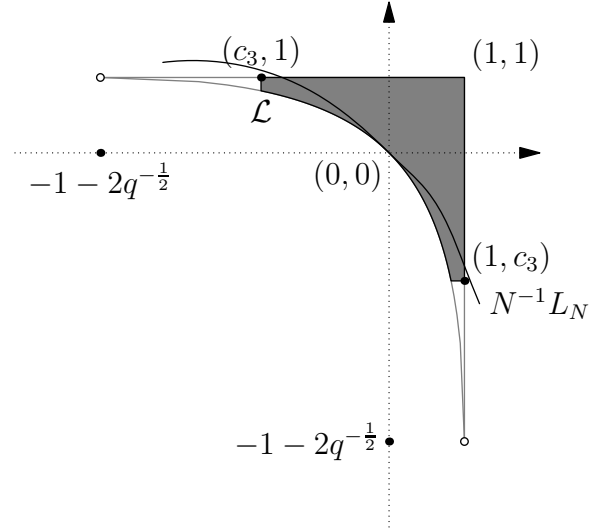


Figure 2: Regions D (shaded) and \tilde{D} (D together with the two corners enclosed by gray lines) and an example of $N^{-1}L_N$.

The following result shows how the Airy process $\mathcal{A}(s)$ (see Section 2.6) arises in describing the spatial fluctuations of point-to-curve LPP. We define the step-like curve L^0 that is approximately an anti-diagonal straight line

$$L^0 = \{(-l^0(s) + s, -l^0(s) - s) \mid s \in \mathbb{R}\}, \quad \text{where} \quad l^0(s) = \begin{cases} k - s & \text{if } s \in [k, k + \frac{1}{2}], \\ s - k - 1 & \text{if } s \in [k + \frac{1}{2}, k + 1]. \end{cases} \quad (15)$$

Proposition 2.3 (Johansson [26]). *Define the stochastic process*

$$H_N(s) := \frac{1}{\mathbf{b}_0 N^{1/3}} \left(\check{G} \left(N + l^0(\mathbf{c}_0 N^{2/3} s) + s \mathbf{c}_0 N^{2/3}, N + l^0(\mathbf{c}_0 N^{2/3} s) - s \mathbf{c}_0 N^{2/3} \right) - \mathbf{a}_0 N \right), \quad (16)$$

where \mathbf{a}_0 is defined in (11) and

$$\mathbf{b}_0 = b_0(1) = \frac{q^{1/6}(1 + \sqrt{q})^{1/3}}{1 - \sqrt{q}}, \quad \text{and} \quad \mathbf{c}_0 = \frac{(1 + \sqrt{q})^{2/3}}{q^{1/6}}. \quad (17)$$

Then on any interval $[-M, M]$, we have the weak convergence (as measures on $\mathcal{C}([-M, M], \mathbb{R})$) as $N \rightarrow \infty$ of

$$H_N(s) \Rightarrow \mathcal{A}(s) - s^2. \quad (18)$$

The definition and some properties of the Airy process are provided in Section 2.6. This functional limit theorem for the fluctuations of all $G_{(N,N)}(-l^0(s) + s, -l^0(s) - s)$ with $s = \mathcal{O}(N^{2/3})$, together with a tightness argument for large s , yields

Proposition 2.4 (Johansson [26]). *As $N \rightarrow \infty$, the point-to-curve last passage time from (N, N) to L^0 satisfies*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{G_{(N,N)}(L^0) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x \right) = \mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - s^2) \leq x \right). \quad (19)$$

2.2 Main result on fluctuations in point-to-curve LPP

Our main result, Theorem 2.6, provides a similar variational characterization as Johansson's results (Proposition 2.4) for point-to-curve LPP with a general class of the lattice paths.

Before stating our theorem, we specify the class of lattice paths which we will consider.

It is clear from (12), (13) and Figure 1 that the horizontal line $y = 1$, the vertical line $x = 1$ and the curve \mathcal{L} enclose a region, which we denote by \tilde{D} . Then we define the region D as the main part of \tilde{D} with the two sharp corners cut off. To be precise, we define, as shown in Figure 2,

$$D = \tilde{D} \setminus \{(x, y) \mid x < c_3 \text{ or } y < c_3\}, \quad \text{where} \quad c_3 \in (-1 - 2q^{-1/2}, 0). \quad (20)$$

The meaning of the constant $-1 - 2q^{-1/2}$ is shown in (14) and Figure 2. Then we let $C \in \mathbb{R}$, $c_1 \in (0, 1)$ and $c_2 \in (0, 1/3)$ be constants, let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $\{m_N\} \subseteq \mathbb{R}_+$ be a sequence of positive real numbers such that

$$\ell(s) < C + c_1 s^2 \quad \text{and} \quad \lim_{N \rightarrow +\infty} m_N = 0. \quad (21)$$

We consider a sequence of lattice paths L_N . For each L_N , we denote its central part as

$$L_N^{\text{central}} = \left\{ (x, y) \in L_N \mid |x - y| < 2\mathbf{c}_0 N^{2/3 + c_2} \right\}. \quad (22)$$

Then we assume the following:

Hypothesis 2.5.

- There is an interval

$$I_N = (a_N, b_N), \quad \text{where} \quad -N^{c_2} \leq a_N < b_N \leq N^{c_2}, \quad (23)$$

and

$$a_N \rightarrow a_\infty \in \{-\infty\} \cup \mathbb{R}, \quad b_N \rightarrow b_\infty \in \{+\infty\} \cup \mathbb{R}, \quad (24)$$

such that

$$L_N^{\text{central}} = \left\{ \left(s\mathbf{c}_0 N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0 N^{1/3}, -s\mathbf{c}_0 N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0 N^{1/3} \right) \mid s \in I_N \right\}, \quad (25)$$

where $l_N(s) : I_N \rightarrow \mathbb{R}$ is a continuous function with

$$\max_{s \in I_N} |l_N(s)| < m_N, \quad (26)$$

and

$$\mathbf{d}_0 = \frac{\mathbf{b}_0}{\mathbf{a}_0} = \frac{(1 + \sqrt{q})^{1/3}}{2q^{1/3}}. \quad (27)$$

- The other part of L_N satisfies

$$\left\{ (x, y) \mid (Nx, Ny) \in L_N \setminus L_N^{\text{central}} \right\} \cap ((-\infty, 1] \times (-\infty, 1]) \subseteq D, \quad (28)$$

as depicted in Figure 2, and

$$\text{dist} \left((L_N \setminus L_N^{\text{central}}) \cap ((-\infty, N] \times (-\infty, N]), \left\{ (x, y) \mid \left(\frac{x}{N}, \frac{y}{N} \right) \in \mathcal{L} \right\} \right) > N^{1/3+2c_2}, \quad (29)$$

where the distance is the Euclidean distance.

Note that we have no requirement of L_N outside of the region $(-\infty, N] \times (-\infty, N]$, because

$$G_{(N,N)}(L_N) = G_{(N,N)}(L_N \cap (-\infty, N] \times (-\infty, N]). \quad (30)$$

Although it suffices to consider $G_{(N,N)}(L_N)$, we state the result with an extra parameter σ to make it parallel to Theorem 2.8 stated later.

Theorem 2.6. Fix $\ell(s)$ and a sequence of lattice paths L_N satisfying Hypothesis 2.5 with constants C, c_1, c_2, c_3 and sequence $\{m_N\}$ defined above Hypothesis 2.5, and also fix $\sigma > 0$. Then for all $\epsilon > 0$ there exists N_0 (depending on $C, c_1, c_2, c_3, \{m_N\}$ and σ but not $\ell(s)$ or L_N) such that for all $N > N_0$ and all $x \in \mathbb{R}$

$$\left| \mathbb{P} \left(\frac{G_{(N+[\sigma\mathbf{c}_0 N^{2/3}], N-[\sigma\mathbf{c}_0 N^{2/3}])}(L_N) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x \right) - \mathbb{P} \left(\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - (s - \sigma)^2 + \ell(s)) \leq x \right) \right| < \epsilon. \quad (31)$$

Note that $\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - (s - \sigma)^2 + \ell(s))$ is a well defined random variable, see Corollary 2.17.

Remark 1. Theorem 2.6, as well as the subsequently stated results of Theorems 2.8 and 2.11 are stated for deterministic initial / boundary data. Here in the statement of the theorem, and later in the proof, we show that the convergence rate is independent of the particular formula of $\ell(s)$ and the particular shape of L_N . This is because later we are going to use the result (actually its analog in TASEP model detailed below) when $\ell(s)$ is random, say, distributed as the path of random walk. Focusing on the above result, assume that $\ell(s)$ is random and that for all $\epsilon > 0$ there exist constants $C \in \mathbb{R}$ and $c_1 \in (0, 1)$ such that with probability at least $1 - \epsilon$, $\ell(s) < C + c_1 s^2$ for all s . Then Theorem 2.6 holds for such a random $\ell(s)$. Instead of coupling all initial data L_N to a single (possibly random) $\ell(s)$ it is also possible to consider L_N which satisfy all of the conditions of Hypothesis 2.5, except that (25) is replaced by

$$L_N^{\text{central}} = \left\{ \left(s\mathbf{c}_0 N^{2/3} - (\ell_N(s) + l_N(s))\mathbf{d}_0 N^{1/3}, -s\mathbf{c}_0 N^{2/3} - (\ell_N(s) + l_N(s))\mathbf{d}_0 N^{1/3} \right) \mid s \in I_N \right\}, \quad (32)$$

where $\ell_N(s)$ converges as a spatial process to some (possibly random) $\ell(s)$ satisfying the aforementioned bounds.

The above theorem is proved in Section 3.1. There are two main ingredient in the proof. The first one is to show that the end of the longest path most likely lies in the vicinity of $(0, 0)$, that is, a point (x, y) with $x + y = \mathcal{O}(N^{1/3})$ and $x - y = \mathcal{O}(N^{2/3})$. The second one is to show that the theorem holds in the special case that L_N is of length $N^{2/3}$, which is proved by the uniform slow decorrelation property of the LPP model, see Theorem 2.18.

2.3 TASEP with general initial data

For the analysis of the TASEP model, we introduce a slightly different LPP model where the i.i.d. random variables $w^*(i, j)$ associated to each site are geometrically distributed on $\mathbb{Z}_{>0}$

$$w^*(i, j) \sim w(i, j) + 1, \quad \text{such that} \quad \mathbb{P}(w^*(i, j) = k) = (1 - q)q^{k-1}, \quad k = 1, 2, \dots \quad (33)$$

We similarly define the point-to-point LPP $G_{(x', y')}^*(x, y)$, point-to-curve LPP $G_{(x', y')}^*(L)$, and the reversed LPP $\check{G}_{(x', y')}^*(x, y)$ by (3), (5) and (4) with the weights changed from $w(i, j)$ to $w^*(i, j)$. They have simple relations to the LPPs $G_{(x', y')}^*(x, y)$, $G_{(x', y')}^*(L)$ and $\check{G}_{(x', y')}^*(x, y)$ defined there, for example, if one of (x, y) and (x', y') is a lattice point,

$$G_{(x', y')}^*(x, y) = G_{(x', y')}^*(x, y) + x' + y' - x - y + 1. \quad (34)$$

The TASEP model considered in our paper is that with discrete time and parallel updating dynamics [9], and is defined as follows. Let infinitely many particles be initially at time $t = 0$ placed on the integer lattice \mathbb{Z} such that no lattice site is occupied by more than one particle, and there are infinitely many particles to the left of 0. At each integer time, the particles decide whether to jump to the right neighboring site simultaneously. For any particle x at time $t = n$, if its right neighboring site $x(n) + 1$ was occupied at $t = n$, then it does not move and $x(n+1) = x(n)$; otherwise it jumps to the right neighboring site ($x(n+1) = x(n) + 1$) with probability $1 - q$, or does not move ($x(n+1) = x(n)$) with probability q .

At any time $t \geq 0$, we represent the positions of the particles by the height function $h(\cdot; t) : \mathbb{R} \rightarrow \mathbb{R}$. We let $h(0; t) = 2N_t$ where N_t is the number of particles that have jumped from site -1 to

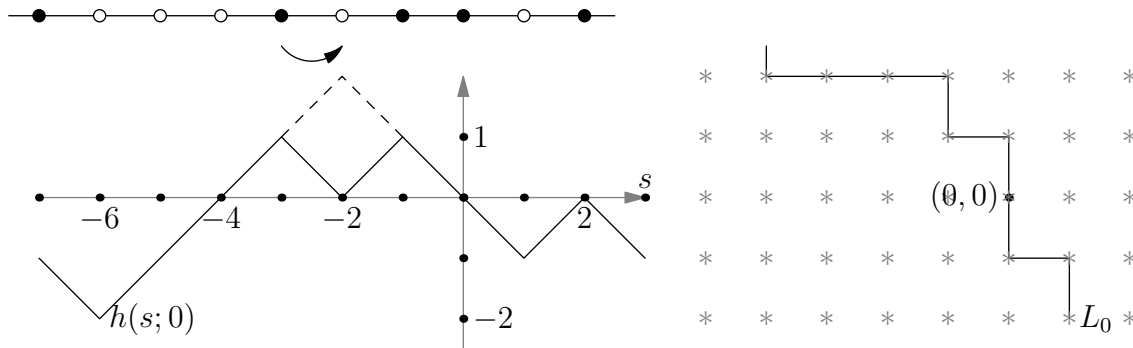


Figure 3: The height function $h(s; t)$ at the initial time $t = 0$. If at time $t = 1$ one particle jumps from -3 to -2 , then $h(s; 1)$ is changed into the dashed shape. The polygonal chain L representing the initial state of the model is shown on the right.

site 0 during the time interval $[0, t)$. For any integer k , we define $h(k; t)$ inductively from $h(0; t)$ by $h(k+1; t) - h(k; t) = \pm 1$ where the sign is positive (negative resp.) if the site k is vacant (occupied resp.) by a particle at time t . At last, for non-integer s , we define $h(s; t)$ by the linear interpolation between $h([s]; t)$ and $h([s] + 1; t)$. See Figure 3 for an example. Also noting that $h(s; t) = h(s; [t])$ for all $t \in \mathbb{R}_+$, we have that $h(s; t)$ is determined by the values of $h(k; n)$ where $k, n \in \mathbb{Z}$. Another observation is that the value of $h(k; t)$ is an integer that has the same parity of k .

To analyze the dynamics of the TASEP model, or equivalently, the dynamics of the height function $h(s; t)$, we introduce the polygonal chain

$$L = \left\{ \left(\frac{s}{2} + \frac{1}{2}h(s; 0), -\frac{s}{2} + \frac{1}{2}h(s; 0) \right) \mid s \in [K_1, K_2] \right\}, \quad (35)$$

to represent the initial configuration of the model, as shown in Figure 3, where K_1 is the position of the leftmost unoccupied site at $t = 0$ if it exists, or $-\infty$ otherwise, and K_2 is one plus the position of the rightmost occupied site at $t = 0$ if it exists, or $+\infty$ otherwise. .

The TASEP model can be coupled to the LPP model with weights $w^*(i, j)$ defined in (33) (see [25, 15] for example). The relation between the distribution of $h(j; t)$ and the LPP is given by

$$\mathbb{P}(h(j; t) > k) = \mathbb{P} \left(G_{\left(\frac{k+j}{2}, \frac{k-j}{2}\right)}^*(L) \leq t \right), \quad (36)$$

for any $j, k \in \mathbb{Z}$ with the same parity. Here L is the polygonal chain defined in (35). This coupling follows by defining the TASEP height function at time t as the rotated envelop of all points which has last passage time less than or equal to t . The weights correspond with the probabilities of particle movement.

2.4 Main result on TASEP with general initial data

Now we consider the TASEP model with general initial condition. Since the TASEP model is mapped to the LPP model with weight function given in (33), the result for the TASEP is analogous to that of the LPP model stated in Section (2.2). Below we set up the notations for the LPP model with weight (33), give technical conditions in terms of LPP, and then present the result in terms of the TASEP model.

Analogous to Proposition 2.1(a), we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \check{G}^*(\gamma N, N) = a_0^*(\gamma) \text{ almost surely, where } a_0^*(\gamma) = a_0(\gamma) + \gamma + 1 = \frac{\gamma + 1 + 2\sqrt{\gamma q}}{1 - q}. \quad (37)$$

Then parallel to $\check{\mathcal{L}}$ and \mathcal{L} defined in (12) and (13), we define

$$\mathbf{a}_0^* = a_0^*(1) = \frac{2}{1 - \sqrt{q}} = \mathbf{a}_0 + 2, \quad (38)$$

and then

$$\begin{aligned} \check{\mathcal{L}}^* &:= \left\{ (x, y) \in (0, \infty) \times (0, \infty) \mid ya_0^* \left(\frac{x}{y} \right) = \mathbf{a}_0^* \right\} \\ &= \left\{ (r(\theta) \cos \theta, r(\theta) \sin \theta) \mid \theta \in (0, \frac{\pi}{2}) \text{ and } r(\theta) = \frac{2(1 + \sqrt{q})}{(\cos \theta + \sin \theta) + 2\sqrt{\cos \theta \sin \theta q}} \right\}, \end{aligned} \quad (39)$$

and

$$\mathcal{L}^* := \{(x, y) \mid (1 - y, 1 - x) \in \check{\mathcal{L}}^*\}. \quad (40)$$

By Theorem 2.1(a), (b) again, we have that if $(x, y) \in \check{\mathcal{L}}^*$, $\check{G}^*(xN, yN) = \mathbf{a}_0^*N + o(N)$, and equivalently if $(x, y) \in \mathcal{L}^*$, $G_{(N, N)}^*(xN, yN) = \mathbf{a}_0^*N + o(N)$. Then parallel to the regions \tilde{D} and D shown in Figure 2, we define the region \tilde{D}^* as the region enclosed by $x = 1$, $y = 1$ and \mathcal{L}^* , and then

$$D^* = \tilde{D}^* \setminus \{(x, y) \mid x < c_3^* \text{ or } y < c_3^*\}, \quad \text{where } c_3^* \in (-1 - 2q^{1/2}, 0), \quad (41)$$

where the value $-1 - 2q^{1/2}$ is analogous to the value $1 - 2q^{-1/2}$ in (20). We also let $C \in \mathbb{R}$, $c_1 \in (0, 1)$, $c_2 \in (0, 1/3)$, let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $\{m_N\} \subseteq \mathbb{R}_+$ be a sequence of positive real numbers such that (21) is satisfied.

We consider a sequence of lattice paths L_N^* analogous to L_N considered in Section 2.2. We assume that each L_N^* is defined by the initial condition of a TASEP model, that is, for each index N , we consider the TASEP model represented by a height function $h_N(s; t)$, and then let L_N^* be the polygonal chain L that is defined by $h_N(s; 0)$ in (35). For each L_N^* , we denote its central part as

$$L_N^{*, \text{central}} = \left\{ (x, y) \in L_N^* \mid |x - y| \leq 2\mathbf{c}_0 N^{2/3 + c_2} \right\}. \quad (42)$$

Then we assume the following

Hypothesis 2.7.

- *There is an interval*

$$I_N^* = (a_N, b_N), \quad (43)$$

where

$$-N^{c_2} \leq a_N < b_N \leq N^{c_2}, \quad \text{and } a_N \rightarrow a_\infty \in \{-\infty\} \cup \mathbb{R}, \quad b_N \rightarrow b_\infty \in \{+\infty\} \cup \mathbb{R}, \quad (44)$$

such that

$$L_N^{*, \text{central}} = \left\{ \left(s\mathbf{c}_0 N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0^* N^{1/3}, -s\mathbf{c}_0 N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0^* N^{1/3} \right) \mid s \in I_N^* \right\}, \quad (45)$$

or equivalently,

$$h_N(2s\mathbf{c}_0N^{2/3}; 0) = -2(\ell(s) + l_N(s))\mathbf{d}_0^*N^{1/3}, \quad s \in I_N^*, \quad (46)$$

where \mathbf{c}_0 is defined in (17), $l_N(s) : I_N \rightarrow \mathbb{R}$ is a continuous function with

$$\max_{s \in I_N} |l_N(s)| < m_N, \quad (47)$$

and \mathbf{d}_0^* is defined, analogous to (27), as

$$\mathbf{d}_0^* = \frac{\mathbf{b}_0}{\mathbf{a}_0^*} = \frac{q^{1/6}(1 + \sqrt{q})^{1/3}}{2}. \quad (48)$$

- The other part of L_N^* satisfies

$$\left\{ (x, y) \mid (Nx, Ny) \in L_N^* \setminus L_N^{*,\text{central}} \right\} \cap ((-\infty, 1] \times (-\infty, 1]) \subseteq D^*, \quad (49)$$

and

$$\text{dist} \left(\left(L_N^* \setminus L_N^{*,\text{central}} \right) \cap ((-\infty, N] \times (-\infty, N]), \left\{ (x, y) \mid \left(\frac{x}{N}, \frac{y}{N} \right) \in \mathcal{L}^* \right\} \right) > N^{1/3+2c_2}, \quad (50)$$

where the distance is the Euclidean distance.

Theorem 2.8. Fix $\ell(s)$ and a sequence of lattice paths L_N^* satisfying Hypothesis 2.7 with constants C, c_1, c_2, c_3^* and sequence $\{m_N\}$ defined above Hypothesis 2.7, and also fix $\sigma > 0$. Here for each N , L_N^* is associated to the initial condition of a TASEP model whose height function is denoted by $h_N(s; t)$ via the relation (35). Then for all $\epsilon > 0$ there exists N_0 (depending on C, c_1, c_2, c_3^* and $\{m_N\}, \sigma$ but not $\ell(s)$ or L_N^*) such that for all $N > N_0$ and all $x \in \mathbb{R}$, the height function $h_N(s; 0)$ of the TASEP model, as defined in (35), satisfies

$$\left| \mathbb{P} \left(\frac{h_N(2\sigma\mathbf{c}_0N^{2/3}; \mathbf{a}_0^*N) - 2N}{2\mathbf{d}_0^*N^{1/3}} > -x \right) - \mathbb{P} \left(\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - (s - \sigma)^2 + \ell(s)) < x \right) \right| < \epsilon. \quad (51)$$

Remark 2. By the relation (36), we have that under the assumption that $2\mathbf{d}_0^*N^{1/3}x$ and $2\sigma\mathbf{c}_0N^{2/3}$ are integers with the same parity,

$$\mathbb{P} \left(\frac{h_N(2\sigma\mathbf{c}_0N^{2/3}; \mathbf{a}_0^*N) - 2N}{2\mathbf{d}_0^*N^{1/3}} > -x \right) = \mathbb{P} \left(G_{(N+\sigma\mathbf{c}_0N^{2/3}-\mathbf{d}_0^*N^{1/3}x, N-\sigma\mathbf{c}_0N^{2/3}-\mathbf{d}_0^*N^{1/3}x)}^*(L_N^*) \leq [\mathbf{a}_0^*N] \right). \quad (52)$$

The above equation implies Theorem 2.8 on TASEP is equivalent to an analog of Theorem 2.6 on LPP.

Below we list several typical initial conditions of TASEP, and their initial height functions. We characterize the initial height function $h(s; 0)$ only at integer-valued s . Note that all the initial conditions are N -independent. Theorem 2.8 covers more general, N -dependent initial conditions, for example, periodic initial conditions with period $\mathcal{O}(N^{2/3})$.

- (Step initial condition) Initially all negative sites are occupied and all non-negative sites are empty, i.e.,

$$h^{\text{step}}(s; 0) = |s|. \quad (53)$$

- (Flat initial condition) Initially all even sites are occupied and all odd sites are empty, i.e.,

$$h^{\text{flat}}(s; 0) = \begin{cases} 0 & \text{if } s = 0, \pm 2, \pm 4, \dots, \\ -1 & \text{if } s = \pm 1, \pm 3, \dots, \end{cases} \quad (54)$$

- (Brownian/Bernoulli/stationary initial condition) Initially all sites are independently occupied with probability $\frac{1}{2}$ and empty with probability $1/2$, i.e.,

$$h^{\text{Bern}}(s; 0) = \begin{cases} 0 & \text{if } s = 0. \\ \sum_{i=0}^{s-1} w_i & \text{if } s = 1, 2, \dots, \\ -\sum_{i=-s}^{-1} w_i & \text{if } s = -1, -2, \dots, \end{cases} \quad (55)$$

and w_i are random variables in i.i.d. two-point distribution such that $\mathbb{P}(w_i = 1) = 1/2$ and $\mathbb{P}(w_i = -1) = 1/2$.

- (Wedge-flat initial condition) Initially all negative sites and all even sites are occupied, but all positive odd sites are empty, i.e.,

$$h^{\text{step/flat}}(s; 0) = \begin{cases} h^{\text{flat}}(s; 0) & \text{if } s \geq 0, \\ h^{\text{step}}(s; 0) & \text{if } s < 0. \end{cases} \quad (56)$$

- (Wedge-Bernoulli initial condition) Initially all negative sites are occupied, and all non-negative sites are independently occupied with probability $1/2$ and empty with probability $1/2$, i.e.,

$$h^{\text{step/Bern}}(s; 0) = \begin{cases} h^{\text{Bern}}(s; 0) & \text{if } s \geq 0, \\ h^{\text{step}}(s; 0) & \text{if } s < 0. \end{cases} \quad (57)$$

- (Flat-Bernoulli initial condition) Initially all even negative sites are occupied, all odd negative sites are empty, and all non-negative sites are independently occupied with probability $1/2$ and empty with probability $1/2$, i.e.,

$$h^{\text{flat/Bern}}(s; 0) = \begin{cases} h^{\text{Bern}}(s; 0) & \text{if } s \geq 0, \\ h^{\text{flat}}(s; 0) & \text{if } s < 0. \end{cases} \quad (58)$$

As consequences of Theorem 2.8 we can prove variation formulas for one-point distributions of TASEP started from initial data as in (54), (55) (56), (57) and (58), we have the following results. To state the results in a uniform way, we denote the two-sided Brownian motion $\mathbf{B}(s)$ by

$$\mathbf{B}(s) = \begin{cases} \mathbf{B}_+(s) & \text{if } s \geq 0, \\ \mathbf{B}_-(-s) & \text{if } s \leq 0, \end{cases} \quad (59)$$

where $\mathbf{B}_+(s)$ and $\mathbf{B}_-(s)$ are independent standard Brownian motions starting at 0.

Corollary 2.9. *Let σ be a real constant and $h(s; t)$ be the height function of the TASEP.*

(a) *With the flat initial condition (54),*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{h^{\text{flat}}(2\sigma \mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) = \mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - s^2) < x \right). \quad (60)$$

(b) *With the Bernoulli initial condition (55),*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{h^{\text{Bern}}(2\sigma \mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) = \mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2}q^{-1/4} \mathbf{B}(s)) < x \right). \quad (61)$$

(c) *With the Wedge-flat initial condition (56),*

$$\lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{h^{\text{step/flat}}(2\sigma \mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) = \mathbb{P} \left(\max_{s \leq \sigma} (\mathcal{A}(s) - s^2) < x \right). \quad (62)$$

(d) *With the Wedge-Bernoulli initial condition (57),*

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{h^{\text{step/Bern}}(2\sigma \mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) \\ = \mathbb{P} \left(\max_{s \geq 0} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2}q^{-1/4} \mathbf{B}(s)) < x \right). \end{aligned} \quad (63)$$

(e) *With the Flat-Bernoulli condition (58),*

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{P} \left(\frac{h^{\text{flat/Bern}}(2\sigma \mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) = \\ \mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2}q^{-1/4} \chi_{s \geq 0} \mathbf{B}(s)) < x \right). \end{aligned} \quad (64)$$

Remark 3. The result for the flat initial condition (54) is obtained in [26] and is given, in an equivalent form, in Proposition 2.4 in the case that $\sigma = 0$. Since the flat initial condition is translational invariant, the result holds for general σ . The step initial condition is singular in the sense that $K_1 = K_2 = 0$ in (35) and hence $a_N = b_N = 0$ in (43) and then the interval (a_∞, b_∞) is degenerate into a point $\{0\}$. The result, which is stated in Proposition 2.1(c), actually is used in the proof of Theorem 2.8, so we do not list it as a corollary. The situation is comparable to that explained in [16, Remark 1.6].

Comparing the results in Corollary 2.9 with the asymptotics of $h(s; t)$ obtained in continuous TASEP models (see [4], [10], [2], [11] for details) that corresponds to the $q \rightarrow 1_-$ limit of the

discrete TASEP model considered in this paper, we obtain, modulo a change in order of taking limits $q \rightarrow 1_-$ and $N \rightarrow \infty$, (which we do not justify it in this paper)

$$\mathbb{P} \left(\max_{t \in \mathbb{R}} (\mathcal{A}(s) - (s - \sigma)^2) < x \right) = \mathbb{P}(2^{1/3} \mathcal{A}_1(2^{-2/3} \sigma) < x), \quad (65)$$

$$\mathbb{P} \left(\max_{t \in \mathbb{R}} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2} \mathbf{B}(s)) < x \right) = \mathbb{P}(\mathcal{A}_{\text{stat}}(\sigma) < x), \quad (66)$$

$$\mathbb{P} \left(\max_{s \geq 0} (\mathcal{A}(s) - (s - \sigma)^2) < x \right) = \mathbb{P}(\mathcal{A}_{2 \rightarrow 1}(\sigma) < x + \sigma^2 \chi_{\sigma < 0}), \quad (67)$$

$$\mathbb{P} \left(\max_{s \geq 0} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2} \mathbf{B}(s)) < x \right) = \mathbb{P}(\mathcal{A}_{\text{BM} \rightarrow 2}(-\sigma) < x + \sigma^2), \quad (68)$$

$$\mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - (s - \sigma)^2 + \sqrt{2} \chi_{s \geq 0} \mathbf{B}(s)) < x \right) = \mathbb{P}(\mathcal{A}_{2 \rightarrow 1, 1, 0}(-\sigma) < x + \sigma^2 \chi_{\sigma > 0}). \quad (69)$$

Below are explanations of notations:

- In (65), \mathcal{A}_1 stands for the Airy process with flat initial data, defined in [34] and [8, Formulas (1.4) and (1.5)]. The \mathcal{A}_1 process is stationary, and its 1-dimensional distribution is [21]

$$\mathbb{P}(\mathcal{A}_1(\sigma) < x) = F_{\text{GOE}}(2x), \quad (70)$$

where F_{GOE} is the Tracy-Widom GOE distribution [36].

- In (66), $\mathcal{A}_{\text{stat}}$ stands for the Airy process with stationary initial data, defined in [4], and we follow the notation in [31, Section 1.11] and [33, Section 1.2]. The 1-dimensional distribution of $\mathcal{A}_{\text{stat}}(\sigma)$ appears also in literature as (see [4, Remark 1.3] and [22, Appendix A])

$$\mathbb{P}(\mathcal{A}_{\text{stat}}(\sigma) < x) = F_{\sigma}(x) = H \left(x + \sigma^2; \frac{\sigma}{2}, -\frac{\sigma}{2} \right), \quad (71)$$

where $F_{\sigma}(x)$ is defined in [22, Formula (1.20)] and $H(x; w_+, w_-)$ is defined in [5, Definition 3].

- In (67), the transition process $\mathcal{A}_{2 \rightarrow 1}$ interpolating the \mathcal{A}_2 and \mathcal{A}_1 processes is introduced in [10, Definition 2.1] (see also [32, Formula (1.7)], where the notation for the right-hand side of (67) is $G_{\sigma}^{2 \rightarrow 1}(x + \sigma^2 \chi_{\sigma < 0})$).
- In (68), the transition process $\mathcal{A}_{\text{BM} \rightarrow 2}$ interpolating the Brownian motion and \mathcal{A}_2 process is introduced in [24, Formula (3.6)], see also [14, Definition 2.13]. The 1-dimensional distribution of $\mathcal{A}_{\text{BM} \rightarrow 2}(\sigma)$ was conjectured in [29] and proved in [7] to be

$$\mathbb{P}(\mathcal{A}_{\text{BM} \rightarrow 2}(\sigma) < x) = F_1(x; \sigma), \quad (72)$$

where the distribution function F_1 is introduced in [2, Definition 1.3].

- In formula (69), The transition process $\mathcal{A}_{2 \rightarrow 1, 1, 0}$ interpolating the Brownian motion and the \mathcal{A}_1 process is introduced in [11, Definition 18]. It is defined from the TASEP with one slow particle, and it is related to the TASEP with flat-Bernoulli initial condition via Burke's theorem, as explained in [11].

Among formulas (65), (66), (67), (68) and (69), (65) is proved in [26], and then proved in a direct way in [19]. Formula (67) is proved in [32]. Formulas (66), (68) and (69) are conjectured in [33, Section 1.4]. Note that in [33, Section 1.4], the notations $\mathcal{A}_{1 \rightarrow BM}$ and $\mathcal{A}_{2 \rightarrow BM}$ are described but not precisely defined. From the context we figure out that

$$\mathcal{A}_{2 \rightarrow BM}(\sigma) = \mathcal{A}_{BM \rightarrow 2}(-\sigma) - \sigma^2 \chi_{\sigma > 0}, \quad \mathcal{A}_{1 \rightarrow BM}(\sigma) = \mathcal{A}_{2 \rightarrow 1,1,0}(-\sigma) - \sigma^2 \chi_{\sigma > 0}. \quad (73)$$

Formulas (66) and (68) are special cases of Corollary 2.15 (c) with $w_+ = -w_- = \frac{\sigma}{2}$ and (a) with $k = 1$ in Section 2.5. And our argument in this paper is a strong support to the conjectural formula (69).

Remark 4. The method in our study of the discrete time TASEP, if applied on the continuous time TASEP, that is, the $q \rightarrow 1_-$ limit of the discrete time one, yields the counterparts of (61), (62), (63) and (64) with $q = 1$, and then the formulas (65), (66), (67), (68) and (69) are derived directly. The only technical obstacle in the application of our method in the continuous time TASEP is that the counterpart of Proposition 2.4, where the discrete geometric distribution of $w(i, j)$ is replaced by the continuous exponential distribution is not available in literature. We remark that the counterpart of Proposition 2.4 can also be proved by the method in [26].

2.5 LPP with an inhomogeneous weight distribution

In this subsection we consider the point-to-point LPP on a \mathbb{Z}^2 lattice where the weights on sites are in independent geometric distribution, but with nonidentical parameters. The strategy is to express the point-to-point LPP with respect to these weights by point-to-curve LPP with respect to homogeneous weight as considered in Section 2.1.

Let L be the vertical path (depending on N which we suppress)

$$L := \{(0, y) \mid y \in D_N\}, \quad \text{where } D_N \text{ is an interval on } \mathbb{R}. \quad (74)$$

We are most interested in the case that $D_N = \mathbb{R}$. But the LPP $G_{(N,N)}(L)$ is not well defined in this case, since $G_{(N,N)}(0, y) \rightarrow +\infty$ almost surely as $y \rightarrow -\infty$. We consider a modified LPP

$$G_{(N,N)}^{f_N}(L) = \max_{y \in D_N} (G_{(N,N)}(0, y) - f_N(y)) \quad (75)$$

where $f_N : D_N \rightarrow \mathbb{R}$ is a function where D_N , the domain of f_N , is an interval. This modified LPP $G_{(N,N)}^{f_N}(L)$ is well defined for $D_N = \mathbb{R}$ if $f_N(x) \rightarrow +\infty$ fast enough as $x \rightarrow -\infty$.

By Proposition 2.1, for $y = cN$ where c is in a compact subset of $(-\infty, 1)$, if $f_N(y) = a_0(1 - y/N)N$, then $G_{(N,N)}(0, y) = o(N)$. So if $f_N(y)$ is close to $a_0N = a_0(1)N$ for y around 0, and otherwise greater than $a_0(1 - y/N)N$ for all $y < N$, then $G_{(N,N)}^{f_N}(L)$ is $o(N)$ and the value of y such that $G_{(N,N)}(0, y) - f_N(y)$ attains its maximum in the vicinity of 0. To make the idea above precise, we state a technical hypothesis for f_N analogous to Hypotheses 2.5 and 2.7. First let $C \in \mathbb{R}$, $c_1 \in (0, 1)$, $c_2 \in (0, 1/3)$ and $c_4 > 0$ be constants, let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $\{m_N\} \subseteq \mathbb{R}_+$ be a sequence of positive numbers such that (21) is satisfied.

Hypothesis 2.10.

- *There is an interval*

$$I_N = (a_N, b_N), \quad (76)$$

$$-N^{c_2} \leq a_N < b_N \leq N^{c_2}, \quad \text{and} \quad a_N \rightarrow a_\infty \in \{-\infty\} \cup \mathbb{R}, \quad b_N \rightarrow b_\infty \in \{+\infty\} \cup \mathbb{R}, \quad (77)$$

such that

$$f_N(2s\mathbf{c}_0N^{2/3}) = \mathbf{a}_0N - s\mathbf{a}_0\mathbf{c}_0N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0N^{1/3}, \quad (78)$$

for all $s \in I_N$, where $l_N(s) : \mathbb{R} \rightarrow \mathbb{R}$ is any continuous function with $\max_{s \in I_N} |l_N(s)| < m_N$, and $\ell(s)$ and m_N are specified in (21).

- For all $y \in D_N$ such that $yN^{1/3}/(2\mathbf{c}_0) \in (-\infty, N^{1/3}/(2\mathbf{c}_0)] \setminus I_N$, $f_N(yN)$ satisfies the inequality

$$\frac{f_N(yN)}{N} > \max \left(\mathbf{a}_0 - \frac{\mathbf{a}_0y}{2} - c_1\mathbf{d}_0 \left(\frac{y}{2\mathbf{c}_0} \right)^2, a_0(1-y) + c_4|y| \right). \quad (79)$$

Theorem 2.11. Fix $\ell(s)$ and a sequence of functions f_N satisfying Hypothesis 2.10 with constants $C, c_1, c_2, c_4, c_5, \{m_N\}$ defined above Hypothesis 2.10. Then for all ϵ there exists N_0 (depending on $C, c_1, c_2, c_4, c_5, m_N$ but not $\ell(s)$ or f_N) such that for all $N > N_0$ and all $x \in \mathbb{R}$,

$$\left| \mathbb{P} \left(\frac{G_{(N,N)}^{f_N}(L)}{\mathbf{b}_0N^{\frac{1}{3}}} < x \right) - \mathbb{P} \left(\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - s^2 + \ell(s)) < x \right) \right| < \epsilon. \quad (80)$$

Another similar question is to consider the \perp -shaped path

$$\tilde{L} = \{(0, y) \mid y \geq 0\} \cup \{(x, 0) \mid x \geq 0\}. \quad (81)$$

The LPP $G_{(N,N)}(\tilde{L})$ is well defined and equivalent to the point-to-point LPP $G_{(N,N)}(0,0)$. If $\tilde{f}_N : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function such that $\tilde{f}_N(x)$ increases at a proper speed as $|x|$ increases, then the modified LPP

$$G_{(N,N)}^{\tilde{f}_N}(\tilde{L}) = \max \left(\max_{y \geq 0} \left(G_{(N,N)}(0, y) - \tilde{f}_N(y) \right), \max_{x \geq 0} \left(G_{(N,N)}(x, 0) - \tilde{f}_N(-x) \right) \right) \quad (82)$$

has nontrivial limiting property like that of $G_{(N,N)}^{f_N}(L)$ stated in Theorem 2.11. To make the idea above precise, we state a technical hypothesis for \tilde{f}_N analogous to Hypothesis 2.10. First let $C \in \mathbb{R}$, $c_1 \in (0, 1)$, $c_2 \in (0, 1/3)$ and $c_4 > 0$ be constants, let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $\{m_N\} \subseteq \mathbb{R}_+$ be a sequence of positive numbers such that (21) is satisfied.

Hypothesis 2.12.

- There is an interval

$$I_N = (a_N, b_N), \quad (83)$$

$$-N^{c_2} \leq a_N < b_N \leq N^{c_2}, \quad \text{and} \quad a_N \rightarrow a_\infty \in \{-\infty\} \cup \mathbb{R}, \quad b_N \rightarrow b_\infty \in \{+\infty\} \cup \mathbb{R}, \quad (84)$$

such that

$$\tilde{f}_N(2s\mathbf{c}_0N^{2/3}) = \mathbf{a}_0N - |s|\mathbf{a}_0\mathbf{c}_0N^{2/3} - (\ell(s) + l_N(s))\mathbf{d}_0N^{1/3}, \quad (85)$$

for all $s \in I_N$, where $l_N(s) : \mathbb{R} \rightarrow \mathbb{R}$ is any continuous function with $\max_{s \in I_N} |l_N(s)| < m_N$ and $\ell(s)$ and m_N are specified in (21).

- For all $yN^{1/3}/(2\mathbf{c}_0) \in [-N^{1/3}/(2\mathbf{c}_0), N^{1/3}/(2\mathbf{c}_0)] \setminus I_N$, $\tilde{f}_N(yN)$ satisfies the inequality

$$\frac{\tilde{f}_N(yN)}{N} > \max \left(\mathbf{a}_0 N - \frac{\mathbf{a}_0 |y|}{2} - c_1 \mathbf{d}_0 \left(\frac{y}{2\mathbf{c}_0} \right)^2, a_0(1 - |y|) + c_4 |y| \right). \quad (86)$$

Theorem 2.13. Fix $\ell(s)$ and a sequence of functions \tilde{f}_N satisfying Hypothesis 2.12 with constants C, c_1, c_2, c_4 and sequence $\{m_N\}$ defined above Hypothesis 2.12. Then for all ϵ there exists N_0 (depending on $C, c_1, c_2, c_4, \{m_N\}$ but not $\ell(s)$ or \tilde{f}_N) such that for all $N > N_0$ and all $x \in \mathbb{R}$,

$$\left| \mathbb{P} \left(\frac{G_{(N,N)}^{\tilde{f}_N}(\tilde{L})}{\mathbf{b}_0 N^{1/3}} < x \right) - \mathbb{P} \left(\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - s^2 + \ell(s)) \right) \right| < \epsilon. \quad (87)$$

As applications of Theorems 2.11 and 2.13 (or adaption of their proofs, see Remark 5), we have the following results for point-to-point LPP with inhomogeneous weight parameters. The weight parameters we will consider differ from the homogeneous ones considered in Section 2.1 in only finitely many columns and/or rows. So we use the same notation $\check{G}(N, N)$ which is defined in (4) and (2), but the weights on some of the lattice points are defined differently. To state the following corollaries, we denote by $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ two independent Airy processes that are the \mathcal{A} described in Section 2.6, and denote by $\mathbf{B}_1, \dots, \mathbf{B}_k$ independent two-sided Brownian motions that are the \mathbf{B} defined in (59).

Corollary 2.14. In the \mathbb{Z}^2 lattice we consider the point-to-point LPP $\check{G}(N, N)$, and denote

$$\tilde{G}_N = \frac{\check{G}(N, N) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}}, \quad (88)$$

where \mathbf{a}_0 and \mathbf{b}_0 are defined in (11) and (17) respectively.

- (a) Suppose the weights $w(i, j)$ are independent and geometrically distributed with parameter $\alpha_{i,j}$ such that $\alpha_{i,j} = 1 - q$ if $i \notin \{1, 2, \dots, k\}$ and

$$\alpha_{i,j} = 1 - \sqrt{q} \left(1 - \frac{2w_i}{\mathbf{d}_0 N^{1/3}} \right) \quad \text{if } i = 1, \dots, k, \quad (89)$$

where $k \in \mathbb{Z}_+$ and $w_1, \dots, w_k \in \mathbb{R}$ are constants. Then

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tilde{G}_N \leq x) = \mathbb{P} \left(\max_{0=s_0 \leq s_1 \leq \dots \leq s_k} \left(\mathcal{A}(s_k) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) - 4 \sum_{i=1}^k w_i (s_i - s_{i-1}) - s_k^2 \right) \leq x \right). \quad (90)$$

- (b) Suppose the weight $w(0, 0)$ is fixed to be 0, the weights $w(i, j)$ are independent and geometrically distributed with parameter $\alpha_{i,j}$ if i, j are not both 0, such that $\alpha_{i,j} = 1 - q$ if i, j are both nonzero, and

$$\alpha_{i,j} = \begin{cases} 1 - \sqrt{q} \left(1 - \frac{2w_+}{\mathbf{d}_0 N^{1/3}} \right) & \text{if } i \geq 1 \text{ and } j = 0, \\ 1 - \sqrt{q} \left(1 - \frac{2w_-}{\mathbf{d}_0 N^{1/3}} \right) & \text{if } i = 0 \text{ and } j \geq 1. \end{cases} \quad (91)$$

where $w_+, w_- \in \mathbb{R}$ are constants. Then

$$\lim_{N \rightarrow \infty} \mathbb{P}(\tilde{G}_N \leq x) = \mathbb{P}\left(\max_{s \in \mathbb{R}} \left(\mathcal{A}(s) + \sqrt{2}\mathbf{B}(s) + 4(w_+1_{s < 0} - w_-1_{s > 0})s - s^2\right) \leq x\right). \quad (92)$$

(c) Suppose the weight $w(i, j)$ are independent and geometrically distributed with parameter $\alpha_{i,j}$ such that $\alpha_{i,j} = 1 - q$ if $j \leq [\alpha n]$ or $j > [\alpha n] + k$, and

$$\alpha_{i,j} = 1 - \sqrt{q} \left(1 - \frac{2w_{j-[\alpha n]}}{\mathbf{d}_0 N^{1/3}}\right) \quad \text{if } j = [\alpha n] + 1, \dots, [\alpha n] + k, \quad (93)$$

where $\alpha \in (0, 1)$, $k \in \mathbb{Z}_+$ and $w_1, \dots, w_k \in \mathbb{R}$ are constants. Then

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{P}(\tilde{G}_N \leq x) = & \mathbb{P}\left(\max_{s_0 \leq s_1 \leq \dots \leq s_k} \left(\alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s_0) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) \right. \right. \\ & \left. \left. + (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \beta)^{-2/3} s_k) - 4 \sum_{i=1}^k w_i (s_i - s_{i-1}) - \frac{s_0^2}{\alpha} - \frac{s_k^2}{1 - \alpha}\right) \leq x\right). \quad (94) \end{aligned}$$

Remark 5. Parts (a) and (b) of Corollary 2.14 are direct consequences of Theorems 2.11 and 2.13 respectively, but Part (c) does not follow these theorems in a straightforward way, although the proofs of the theorems can be adapted to prove Part (c).

The limits on the left-hand sides of (90), (92) and (94) have been analyzed previously in [5], [2] and [1], and the results were given in other forms by Fredholm determinants. Utilizing these earlier results we arrive at the following expressions for these statistics.

Corollary 2.15. For all $x \in \mathbb{R}$,

(a) for all parameters $w_1, \dots, w_k \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{0=s_0 \leq s_1 \leq \dots \leq s_k} \left(\mathcal{A}(s_k) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) - 4 \sum_{i=1}^k w_i (s_i - s_{i-1}) - s_k^2\right) \leq x\right) = F_k^{\text{spiked}}(x; 2w_1, \dots, 2w_k), \quad (95)$$

(b) for all parameters $\alpha \in (0, 1)$ and $w_1, \dots, w_k \in \mathbb{R}$,

$$\begin{aligned} \mathbb{P}\left(\max_{s_0 \leq s_1 \leq \dots \leq s_k} \left(\alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s_0) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) + (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \beta)^{-2/3} s_k) \right. \right. \\ \left. \left. - 4 \sum_{i=1}^k w_i (s_i - s_{i-1}) - \frac{s_0^2}{\alpha} - \frac{s_k^2}{1 - \alpha}\right) \leq x\right) = F_k^{\text{spiked}}(x; 2w_1, \dots, 2w_k), \quad (96) \end{aligned}$$

(c) for all parameters $w_+, w_- \in \mathbb{R}$,

$$\mathbb{P}\left(\max_{s \in \mathbb{R}} \left(\mathcal{A}(s) + \sqrt{2}\mathbf{B}(s) + 4(w_+1_{s < 0} - w_-1_{s > 0})s - s^2\right) \leq x\right) = H(x; w_+, w_-), \quad (97)$$

where $F_k^{\text{spiked}}(x; w_1, \dots, w_n)$ is the distribution introduced in [2, Formula (54)] and [1, Corollary 1.3], and $H(x; w_+, w_-)$ is the distribution function introduced in [5].

2.6 The Airy process

The Airy process $\mathcal{A}(\cdot)$ [30] (sometimes also denoted as $\mathcal{A}_2(\cdot)$ and called the Airy₂ process, in contrast to the Airy₁ process \mathcal{A}_1 considered in (65)) is an important process appearing in the Kardar-Parisi-Zhang universality class, see for example [13]. Its properties have been intensively studied, see for example [26], [17], [33].

The Airy process $\mathcal{A}(\cdot)$ is defined through its finite-dimensional distributions which are given by a Fredholm determinant formula. For $x_0, \dots, x_n \in \mathbb{R}$ and $t_0 < \dots < t_n$ in \mathbb{R} ,

$$\mathbb{P}(\mathcal{A}(t_0) \leq x_0, \dots, \mathcal{A}(t_n) \leq x_n) = \det(I - f^{1/2} K_{\text{ext}} f^{1/2})_{L^2(\{t_0, \dots, t_n\} \times \mathbb{R})}, \quad (98)$$

where we have counting measure on $\{t_0, \dots, t_n\}$ and Lebesgue measure on \mathbb{R} , f is defined on $\{t_0, \dots, t_n\} \times \mathbb{R}$ by $f(t_j, x) = \mathbf{1}_{x \in (x_j, \infty)}$, and the *extended Airy kernel* [30] is defined by

$$K_{\text{ext}}(t, \xi; t', \xi') = \begin{cases} \int_0^\infty d\lambda e^{-\lambda(t-t')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda), & \text{if } t \geq t' \\ -\int_{-\infty}^0 d\lambda e^{-\lambda(t-t')} \text{Ai}(\xi + \lambda) \text{Ai}(\xi' + \lambda), & \text{if } t < t', \end{cases}$$

where $\text{Ai}(\cdot)$ is the Airy function. It is readily seen that the Airy process is stationary. The one point distribution of \mathcal{A} is the F_{GUE} distribution (i.e., the GUE Tracy-Widom distribution [36]).

Since our main results appear as variational problems involving the Airy process, it is important to know that these problems are well-posed with finite answers. It was proved in [30, Theorem 4.3] and [26, Theorem 1.2] that there exists a measure on $\mathcal{C}(\mathbb{R}, \mathbb{R})$ (continuous functions from $\mathbb{R} \rightarrow \mathbb{R}$ endowed with the topology of uniform convergence on compact subsets) whose finite dimensional distributions coincide with those of the Airy process (i.e., there exists a continuous version of the Airy process). Further properties of the Airy process were demonstrated in [17]. We summarize those properties which we will appeal to. Part (a) of Proposition 2.16 is a special case of [17, Proposition 4.1], (our $\mathcal{A}(t)$ is their $\mathcal{A}_1(t)$), while Part (b) is a generalization of [17, Proposition 4.4] where the parameter c is taken as 1, and the proof can be used for our generalized case with little modification.

Proposition 2.16. (a) *(Local Brownian absolute continuity) For any $s, t \in \mathbb{R}$, $t > 0$, the measure on functions from $[0, t] \rightarrow \mathbb{R}$ given by $\mathcal{A}(\cdot + s) - \mathcal{A}(s)$ is absolutely continuous with respect to Brownian motion of diffusion parameter 2.*

(b) *For all positive constants α and c such that $\alpha < c$, there exists $\epsilon > 0$ and $C(\alpha, c) > 0$ such that for all $t \geq C(\alpha, c) > 0$ and $x \geq -\alpha t^2$,*

$$\mathbb{P}\left(\sup_{s \notin [-t, t]} (\mathcal{A}(s) - cs^2) > x\right) \leq e^{-\epsilon(ct^2 + x)^{3/2}}. \quad (99)$$

One direct consequence of Proposition 2.16 is the well-definedness of the limit distributions in Theorems 2.6, 2.8, 2.11 and 2.13.

Corollary 2.17. *Let $\ell : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function that satisfies (21) and (a_∞, b_∞) be an interval such that $-\infty \leq a_\infty < b_\infty \leq +\infty$. Then $\max_{s \in (a_\infty, b_\infty)} (\mathcal{A}(s) - (s - \sigma)^2 + \ell(s))$ is a well defined random variable.*

The definition of the Airy process given by (98) is not well adapted to studying variational problems (as it only deals with finite dimensional distributions). Let us note that [19, Theorem 2] provides a concise Fredholm determinant formula for $\mathbb{P}(\mathcal{A}(s) \leq g(s) \text{ for } s \in [\ell, r])$, for any interval $[\ell, r]$ and any $g \in H^1([\ell, r])$ (i.e. both g and its derivative are in $L^2([\ell, r])$). As we do not utilize this formula, we do not restate it here.

2.7 Main technical tools

The main technical tools in this paper are results stemming from the uniform slow decorrelation property that allows us to generalize Proposition 2.3 by Johansson, and the Gibbs property of a multilayer line ensemble extension of the LPP model. As this will require some explanation, we delay a discussion of it until Section 6.

Recall the stochastic process $H_N(s)$ defined in (16). We define more generally

$$\tilde{H}_N(s) = \frac{1}{\mathbf{b}_0 N^{1/3}} \left(\check{G} \left(N + \ell_N(s) N^\alpha + s \mathbf{c}_0 N^{2/3}, N + \ell_N(s) N^\alpha - s \mathbf{c}_0 N^{2/3} \right) - \mathbf{a}_0(N + \ell_N(s) N^\alpha) \right), \quad (100)$$

where $\alpha \in [0, 1)$ is a parameter and $\ell_N(s)$ is a sequence of continuous functions such that the curve $L = (\ell_N(s) N^\alpha + s \mathbf{c}_0 N^{2/3}, \ell_N(s) N^\alpha - s \mathbf{c}_0 N^{2/3})$, $(s \in \mathbb{R})$ is a lattice path.

If $\alpha = 0$ and $\ell_N(s) = l^0(s/(\mathbf{c}_0 N^{2/3}))$ where $l^0(s)$ is defined in (15), then $\tilde{H}(s)$ is equal to $H_N(s)$ defined in (16).

Theorem 2.18. *Let $\tilde{H}_N(s)$ be defined in (100) with $\alpha \in (0, 1)$ and $\ell_N(t)$ continuous on $[-M, M]$ and $\max_{s \in [-M, M]} |\ell_N(s)| < C$ for all large enough N . Then $\tilde{H}_N(s) - H_N(s)$ converges in probability to 0 in $\mathcal{C}([-M, M], \mathbb{R})$, that is, given $\epsilon, \delta > 0$, there is an integer N_0 that depends only on M, α and C such that*

$$\mathbb{P} \left(\max_{s \in [-M, M]} |H_N(s) - \tilde{H}_N(s)| \geq \delta \right) < \epsilon \quad (101)$$

if $N > N_0$.

The slow decorrelation property is a common feature in many models in the KPZ universality, including the LPP model, and equivalently the TASEP model, considered in this paper. As a pointwise property, it is studied first in [20] and then comprehensively in [15]. Let $M \rightarrow 0_+$, then we have the result that as $N \rightarrow \infty$, $N^{-1/3}(\check{G}(N, N) - \mathbf{a}_0 N)$ is equal to $N^{-1/3}(\check{G}(N + \ell_N(0) N^\alpha, N + \ell_N(0) N^\alpha) - \mathbf{a}_0(N + \ell_N(0) N^\alpha))$ in probability. This is a special case of the slow decorrelation result obtained in [15], where the characteristic line is the $\pi/4$ radial line. Theorem 2.18 generalizes the pointwise slow decorrelation to be uniform on an interval.

Theorem 2.18 gives control of $\tilde{H}_N(s)$ in any fixed interval $[-M, M]$. Outside this fixed interval we need the following lemma to control the point-to-curve LPPs by point-to-point LPPs as shown in Figure 4. The lemma is a special consequence of the Gibbs property (see Section 6), but it suffices for our paper.

Lemma 2.19. *Suppose $N > 0$, $K_1 < K_2 < K_3$ are integers between $-N$ and N , and M_1, M_2, M_3 are real numbers such that $(K_1, M_1), (K_2, M_2), (K_3, M_3)$ are colinear, i.e.,*

$$\frac{M_1 - M_2}{K_1 - K_2} = \frac{M_2 - M_3}{K_2 - K_3}. \quad (102)$$

Let $c \in (0, 1)$ be a constant and let $l^0(s)$ be defined in (15). Then

$$\begin{aligned} & \mathbb{P} \left(\max_{K_1 \leq s \leq K_2 - c(K_2 - K_1)} \check{G}(N + l^0(s) + s, N + l^0(s) - s) \geq M_0 \right) \\ & \leq (2 + \epsilon_{\min(c(K_2 - K_1), K_3 - K_2)}) \mathbb{P}(\check{G}(N + K_2, N - K_2) \geq M_2) \\ & \quad + \mathbb{P}(\check{G}(N + K_3, N - K_3) \leq M_3), \end{aligned} \quad (103)$$

where for all $t > 0$, ϵ_t is a positive constant such that $\epsilon_t \rightarrow 0$ as $t \rightarrow \infty$.

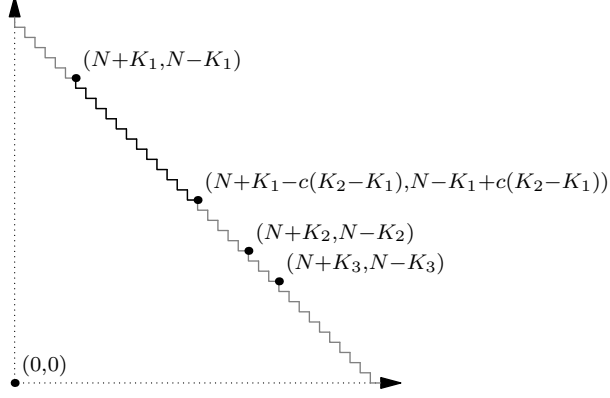


Figure 4: The points $(N + K_1, N - K_1)$, $(N + K_2, N - K_2)$ and $(N + K_3, N - K_3)$ are on the same diagonal lattice path. The left-hand side of (103) is the LPP between the point $(0, 0)$ and the lattice path between $(N + K_1, N - K_1)$ and $(N + K_1 - c(K_2 - K_1), N - K_1 + c(K_2 - K_1))$, shown in solid.

3 Proof of Theorems 2.6, 2.8, 2.11 and 2.13

In this section, we give the detail of the proof of Theorem 2.6 in Section 3.1, and show briefly that Theorem 2.8 can be proved by the method the same as the proof of Theorem 2.6 in Section 3.2. The proofs of Theorems 2.11 and 2.13, (as well as the proof of Corollary 2.14(b)) are by the same method with some adaptations, and we discuss it in Section 3.3.

3.1 Proof of Theorem 2.6

By the translational invariance of the lattice, we can shift the point $(N + [\sigma c_0 N^{2/3}], N - [\sigma c_0 N^{2/3}])$ into (N, N) , and thus if we can prove Theorem 2.6 in the special case that $\sigma = 0$, the general case is proved by shifting the lattice. Therefore, we only prove the $\sigma = 0$ case of Theorem 2.6 for notational simplicity.

In the proof of Theorem 2.6, we suppose x is a fixed real number and the constants C, c_1, c_2, c_3 defined in (21)–(29) are fixed. Without loss of generality, we prove only for the case that I_N , the interval defined in Hypothesis 2.5, is $[-N^{c_2}, N^{c_2}]$. By (30), we only need to consider the curve $(L_N \cap (-\infty, N) \times (-\infty, N))$ where L_N is defined in (21)–(29), and we divide it into parts $L_N^{\text{micro}}(M)$, $L_N^{\text{meso},L}(M)$, $L_N^{\text{meso},R}(M)$, and L_N^{macro} , where the first three depending on a constant $M > 0$, such that, recalling that L_N^{central} defined in (22),

$$L_N^{\text{micro}}(M) = \{(x, y) \in L_N^{\text{central}} \mid |x - y| \leq 2Mc_0N^{2/3}\}, \quad (104)$$

$$L_N^{\text{meso},L}(M) = \{(x, y) \in L_N^{\text{central}} \setminus L_N^{\text{micro}}(M) \mid x < 0\}, \quad (105)$$

$$L_N^{\text{meso},R}(M) = \{(x, y) \in L_N^{\text{central}} \setminus L_N^{\text{micro}}(M) \mid x > 0\}, \quad (106)$$

$$L_N^{\text{macro}} = (L_N \cap (-\infty, N) \times (-\infty, N)) \setminus L_N^{\text{central}}. \quad (107)$$

In Subsection 3.1.1, we show that for any fixed $M > 0$ and $\epsilon > 0$,

$$\left| \mathbb{P} \left(\frac{G_{(N,N)}(L_N^{\text{micro}}(M)) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{1/3}} \leq x \right) - \mathbb{P} \left(\max_{s \in [-M, M]} \mathcal{A}(s) - s^2 + \ell(s) \leq x \right) \right| < \epsilon \quad (108)$$

for all N large enough, independent of the particular formula of $\ell(s)$. In Subsection 3.1.2, we show that for any fixed $\epsilon > 0$, there is an M such that for all N large enough, independent of the particular formula of $\ell(s)$,

$$\mathbb{P} \left(\frac{G_{(N,N)}(L_N^{\text{meso},*}(M)) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{1/3}} > x \right) < \epsilon, \quad \text{for } * = L \text{ or } R, \quad (109)$$

and for any fixed $\epsilon > 0$, for all N large enough, independent of the particular formula of $\ell(s)$,

$$\mathbb{P} \left(\frac{G_{(N,N)}(L_N^{\text{macro}}) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{1/3}} > x \right) < \epsilon. \quad (110)$$

Thus by the three inequalities (108), (109) and (110), and the limit identity

$$\lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) < x \right) = \mathbb{P} \left(\max_{s \in \mathbb{R}} (\mathcal{A}(s) - s^2 + \ell(s)) < x \right) \quad (111)$$

that is a consequence of Proposition 2.16(b), we prove the inequality (31) of Theorem 2.6.

3.1.1 Microscopic estimate

In this subsection we prove that the inequality (108) holds for large enough N , where $M > 0$ and $\epsilon > 0$ is a constant.

The main technique to prove (108) is Theorem 2.18. Since Theorem 2.18 requires a boundedness of $\ell_N(s)$, we first prove (108) under the condition

$$\max_{s \in [-M, M]} |\ell(s)| < N^{\epsilon_2}. \quad (112)$$

Recall the stochastic processes $H_N(s)$ defined in (16), and $\tilde{H}_N(s)$ defined in (100). By the symmetry of the lattice, we have that

$$\mathbb{P} \left(\frac{G_{(N,N)}(L_N^{\text{micro}}(M)) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{1/3}} \leq x \right) = \mathbb{P} \left(\max_{s \in [-M, M]} (\tilde{H}_N(s) + (\ell(s) + l_N(s))) \leq x \right), \quad (113)$$

where $\tilde{H}_N(s)$ is defined with $\ell_N(s) = \ell(s) + l_N(s)$ and the parameter $\alpha = 1/3$.

Since $H_N(s)$, as a stochastic process in $s \in [-M, M]$, converges weakly to $\mathcal{A}(s) - s^2$ and $l_N(s)$ uniformly converges to 0, with the help of Skorohod's representation theorem, we have that for any $\epsilon > 0$ there is a $\delta > 0$ such that for large enough N independent of $\ell(s)$

$$\mathbb{P} \left(\max_{s \in [-M, M]} (H_N(s) + (\ell(s) + l_N(s))) \leq x + \frac{\delta}{2} \right) < \mathbb{P} \left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x + \delta \right) + \frac{\epsilon}{3}, \quad (114)$$

$$\mathbb{P} \left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x - \delta \right) - \frac{\epsilon}{3} < \mathbb{P} \left(\max_{s \in [-M, M]} (\tilde{H}_N(s) + (\ell(s) + l_N(s))) \leq x - \frac{\delta}{2} \right). \quad (115)$$

By Proposition 2.16(a), the Airy process is locally like the Brownian motion [17], so if δ is small enough, then

$$\mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x + \delta\right) - \mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x\right) < \frac{\epsilon}{3}, \quad (116)$$

$$\mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x\right) - \mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x - \delta\right) < \frac{\epsilon}{3}. \quad (117)$$

The uniform slow decorrelation of LPP given in Theorem 2.18 implies that

$$\mathbb{P}\left(\max_{s \in [-M, M]} \left| \tilde{H}_N(s) - H_N(s) \right| > \frac{\delta}{2}\right) < \frac{\epsilon}{3} \quad (118)$$

for large enough N , independent of the particular formula of $\ell(s)$ and $l_N(s)$, as long as $\ell(s)$ satisfies (112).

The inequalities (116), (117), (114), (114) and (118) yield (108) for all $l(s)$ satisfying both (21) and (112). For large enough N independent of $l(s)$,

$$\begin{aligned} \mathbb{P}\left(\frac{G_{(N,N)}(L_N^{\text{micro}}(M)) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{\frac{1}{3}}} \leq x\right) &= \mathbb{P}\left(\max_{s \in [-M, M]} \left(\tilde{H}_N(s) + (\ell(s) + l_N(s))\right) \leq x\right) \\ &< \mathbb{P}\left(\max_{s \in [-M, M]} (H_N(s) + (\ell(s) + l_N(s))) \leq x + \frac{\delta}{2}\right) + \frac{\epsilon}{3} \\ &< \mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + l(s)) \leq x + \delta\right) + \frac{2\epsilon}{3} \\ &< \mathbb{P}\left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + l(s)) \leq x\right) + \epsilon. \end{aligned} \quad (119)$$

Thus one direction of inequality (108) is proved under the condition (112). The proof of the other direction of (108) under the condition (112) is similar.

Finally note that the condition (21) implies (112) for large N . Therefore (108) holds for all $\ell(s)$ that satisfy (21).

3.1.2 Macroscopic and mesoscopic estimates

Macroscopic estimate Inequality (110) is a direct consequence of Lemma 2.2. For any (x, y) on L_N^{macro} . Since $(N^{-1}x, N^{-1}y) \in D$ by (28) where D is defined in (20), and L_N satisfies the relation (29), by Lemma 2.2, we have that for all N large enough,

$$\mathbb{P}\left(\frac{G_{(N,N)}(x, y) - \mathbf{a}_0 N}{\mathbf{d}_0 N^{1/3}} > x\right) < e^{-cN^{2c_2}}, \quad (120)$$

where $c > 0$ depends on c_3 in (29) but not the shape of L_N^{macro} . Note that $G_{(N,N)}(x, y)$ is a constant for (x, y) in a lattice square, and there are fewer than $4(1 + q^{-1/2})^2 N^2$ lattice squares whose image under the scaling transform $(x, y) \rightarrow (N^{-1}x, N^{-1}y)$ is in D . Then we can pick (x_i, y_i) on L^{macro}

where $i = 1, \dots, [4(1+q^{-1/2})^2 N^2]$, such that for all $(x, y) \in L^{\text{macro}}$, $G_{(N,N)}(x, y)$ is equal to at least one $G_{(N,N)}(x_i, y_i)$. Thus

$$\begin{aligned} \mathbb{P}\left(\frac{G_{(N,N)}(L_N^{\text{macro}}) - a_0 N}{d_0 N^{1/3}} > x\right) &< \sum_{i=1}^{[4(1+q^{-1/2})^2 N^2]} \mathbb{P}\left(\frac{G_{(N,N)}(x_i, y_i) - a_0 N}{d_0 N^{1/3}} > x\right) \\ &< 4(1+q^{-1/2})^2 N^2 e^{-cN^{2c_2}}, \end{aligned} \quad (121)$$

and obtain inequality (110) if N is large enough.

Mesoscopic estimate By the symmetry of the lattice model, we only need to prove (109) with $* = R$.

Before giving the proof, we remark that the simple approach in the macroscopic estimate fails in this case, since summing up all the point-to-point LPP between (N, N) and lattice points on $L_N^{\text{meso},R}(M)$ gives a too large upper bound of the point-to-curve LPP $G_{(N,N)}(L_N^{\text{meso},R})$. Before giving the technical proof, we explain the idea. We divide $L_N^{\text{meso},R}$ into segments according to the intervals $I(k)$ in (125). Then on each segment, we estimate the point-to-curve LPP (actually the upper bound $\mathbb{P}(k)$ defined in (127)) by the point-to-point LPPs between $(0, 0)$ and the two points in (131b) and (131c). We estimate the point-to-point LPPs by Lemma 2.2, and the relation between point-to-point LPPs and the point-to-curve LPP is established by Lemma 2.19.

Recall that $L_N^{\text{meso},R}(M) \subseteq L_N^{\text{central}}$ is defined in (25) by a continuous function $\ell(s) + l_N(s)$ for $s \in [M, N^{c_2}]$, where $\ell(s)$ is bounded below by $C + c_1 s^2$ and $l_N(s)$ converges uniformly to 0 as $N \rightarrow \infty$. By the inequality (21), we have that

$$\ell(s) < c'_1 s^2 \quad \text{for all } s \in [\tilde{M}, N^{c_2}], \text{ where } c'_1 \in (c_1, 1) \text{ and } \tilde{M} = \sqrt{C/(c'_1 - c_1)}. \quad (122)$$

Then we take

$$c''_1 \in \left(1, \frac{2}{1 + c'_1}\right). \quad (123)$$

Since x is a constant, it suffices to prove the inequality

$$\mathbb{P}\left(G_{(N,N)}(L_N^{\text{meso},R}(M)) > \mathbf{a}_0 N - c'_1 (c''_1)^2 M^2 \mathbf{d}_0 N^{1/3}\right) < \epsilon \quad (124)$$

for all $M > \tilde{M}$ and large enough N .

For all $k = 0, 1, 2, \dots$ we denote

$$c(k) = (c''_1)^k, \quad C_k = c'_1 (c(k)M)^2, \quad \text{and the interval } I(k) = [c(k-1)M, c(k)M], \quad (125)$$

and define the lattice paths

$$L(k) = \{(s\mathbf{c}_0 N^{2/3} - l_0(s\mathbf{c}_0 N^{2/3}) - [C_k \mathbf{d}_0 N^{1/3}], -s\mathbf{c}_0 N^{2/3} - l_0(s\mathbf{c}_0 N^{2/3}) - [C_k \mathbf{d}_0 N^{1/3}]) \mid s \in I(k)\}. \quad (126)$$

Since on each $I(k)$, $\ell(s) < C_k$ as long as $\ell(s)$ is defined, and $c'_1 (c''_1)^2 M^2 < C_k$ for all k , it is clear that if we denote

$$\mathbb{P}(k) = \mathbb{P}\left((G_{(N,N)}(L(k)) \geq \mathbf{a}_0 N - C_k \mathbf{d}_0 N^{1/3})\right), \quad (127)$$

then as N is large enough,

$$\begin{aligned}
& \mathbb{P} \left(G_{(N,N)}(L_N^{\text{meso},R}(M)) > \mathbf{a}_0 N - c'_1 (c'_1)''^2 M^2 \mathbf{d}_0 N^{1/3} \right) \\
& \leq \mathbb{P} \left(\max_{1 \leq k \leq \lfloor \log N^{c_2} / \log c_1 \rfloor} (G_{(N,N)}(L(k)) \geq \mathbf{a}_0 N - C_k \mathbf{d}_0 N^{1/3}) \right) \\
& \leq \sum_{k=1}^{\lfloor \log N^{c_2} / \log c_1 \rfloor} \mathbb{P}(k).
\end{aligned} \tag{128}$$

To estimate $\mathbb{P}(k)$, we note that by the choice of c'_1 in (123), there exist $\delta_1, \delta_2, \delta_3, \delta_4 > 0$ such that $\delta_2 < \delta_3$ and the points

$$(1, c'_1 (c'_1)''^2), \quad (c'_1 + \delta_1, (1 - \delta_3)(c'_1 + \delta_1)^2), \quad (c'_1 + \delta_2, (1 + \delta_4)(c'_1 + \delta_2)^2) \tag{129}$$

are collinear. Then by a simple affine transformation, the points

$$\begin{aligned}
& \left(N + c(k-1)M\mathbf{c}_0 N^{2/3}, \mathbf{a}_0 N - C_k \mathbf{d}_0 N^{1/3} \right), \\
& \left(N + (c(k) + \delta_1 c(k-1))M\mathbf{c}_0 N^{2/3}, \mathbf{a}_0 N - (1 - \delta_3)(1 + \delta_2/c'_1)''^2 C_k \mathbf{d}_0 N^{1/3} \right), \\
& \left(N + (c(k) + \delta_2 c(k-1))M\mathbf{c}_0 N^{2/3}, \mathbf{a}_0 N - (1 + \delta_4)(1 + \delta_2/c'_1)''^2 C_k \mathbf{d}_0 N^{1/3} \right)
\end{aligned} \tag{130}$$

are collinear, as well as the three points

$$\left(N + [c(k-1)M\mathbf{c}_0 N^{2/3}], \mathbf{a}_0 N - C_k \mathbf{d}_0 N^{1/3} \right), \tag{131a}$$

$$\left(N + [(c(k) + \delta_1 c(k-1))M\mathbf{c}_0 N^{2/3}], \mathbf{a}_0 N - (1 - \delta_{3,N,k})(1 + \delta_1/c'_1)''^2 C_k \mathbf{d}_0 N^{1/3} \right), \tag{131b}$$

$$\left(N + [(c(k) + \delta_2 c(k-1))M\mathbf{c}_0 N^{2/3}], \mathbf{a}_0 N - (1 + \delta_4)(c(k) + \delta_2)^2 M^2 \mathbf{d}_0 N^{1/3} \right) \tag{131c}$$

collinear, where $\delta_{3,N,k} \rightarrow \delta_3$ as $N \rightarrow \infty$ uniformly in k . We only need that for N large enough

$$\delta_{3,N,k} > \frac{\delta_3}{2}. \tag{132}$$

Then by using the symmetry of the lattice and applying Lemma 2.19, we have

$$\begin{aligned}
\mathbb{P}(k) & \leq \mathbb{P} \left(\max_{s=[c(k-1)M\mathbf{c}_0 N^{2/3}] }^{[c(k)M\mathbf{c}_0 N^{2/3}+1]} \check{G}(N + [C_k \mathbf{d}_0 N^{1/3}] + s, N + [C_k \mathbf{d}_0 N^{1/3}] - s) \geq \mathbf{a}_0 N - c'_1 (c(k)M)^2 \mathbf{d}_0 N^{1/3} \right) \\
& \leq (2 + \epsilon_{\min(\delta_1, \delta_2 - \delta_1) \cdot c(k-1)M\mathbf{c}_0 N^{2/3}}) \mathbb{P} \left(\check{G} \left(N + [C_k \mathbf{d}_0 N^{1/3}] + [(c(k) + \delta_1 c(k-1))M\mathbf{c}_0 N^{2/3}], \right. \right. \\
& \quad \left. \left. N + [C_k \mathbf{d}_0 N^{1/3}] + [(c(k) + \delta_1 c(k-1))M\mathbf{c}_0 N^{2/3}] \right) \geq \mathbf{a}_0 N - (1 - \delta_{3,N,k})(1 + \delta_1/c'_1)''^2 C_k \mathbf{d}_0 N^{1/3} \right) \\
& \quad + \mathbb{P} \left(\check{G} \left(N + [C_k \mathbf{d}_0 N^{1/3}] + [(c(k) + \delta_2 c(k-1))M\mathbf{c}_0 N^{2/3}], \right. \right. \\
& \quad \left. \left. N + [C_k \mathbf{d}_0 N^{1/3}] + [(c(k) + \delta_2 c(k-1))M\mathbf{c}_0 N^{2/3}] \right) \leq \mathbf{a}_0 N - (1 + \delta_4)(c(k) + \delta_2/c'_1)''^2 C_k \mathbf{d}_0 N^{1/3} \right),
\end{aligned} \tag{133}$$

where the term $\epsilon_{\min(\delta_1, \delta_2 - \delta_1) \cdot c(k-1) M \mathbf{c}_0 N^{\frac{2}{3}}}$ is defined in Lemma 2.19 and vanishes as $N \rightarrow \infty$. An application of Lemma 2.2, shows that

$$\mathbb{P}(k) < e^{-Mk} \quad (134)$$

for large enough M . Thus (124) is proved by taking the sum of $\mathbb{P}(k)$ in (128).

3.2 Proof of Theorem 2.8

Using the representation of TASEP by the LPP model, and Remark 2 in particular, to prove Theorem 2.8 we need only to compute the $N \rightarrow \infty$ limit of

$$\mathbb{P} \left(G_{(N+\sigma \mathbf{c}_0 N^{2/3} - \mathbf{d}_0^* N^{1/3, x}, N - \sigma \mathbf{c}_0 N^{2/3} - \mathbf{d}_0^* N^{1/3, x})}^*(L_N^*) \leq [\mathbf{a}_0^* N] \right). \quad (135)$$

Thus Theorem 2.8 can be proved by the same method as the proof of Theorem 2.6, since the estimate of the LPP associated to the TASEP differs from the LPP model in Theorem 2.6 only by a constant shift at each lattice site. In fact, the theorem follows as a corollary of Theorem 2.6.

3.3 Proof of Theorems 2.11 and 2.13

For the proof of Theorem 2.11, we assume $D_N = \mathbb{R}$ without loss of generality. We express

$$G_{(N,N)}^{f_N}(L) = \max \left(G_{(N,N)}^{f_N, \text{micro}}(L), G_{(N,N)}^{f_N, \text{meso}}(L), G_{(N,N)}^{f_N, \text{macro}}(L) \right), \quad (136)$$

where

$$G_{(N,N)}^{f_N, *}(L) = \max_{y \in I_*} (G_{(N,N)}(0, y) - f_N(y)), \quad * = \text{micro, meso or macro}, \quad (137)$$

and, letting $M > 0$,

$$I_* = \begin{cases} [-M2\mathbf{c}_0 N^{2/3}, M2\mathbf{c}_0 N^{2/3}] & \text{for } * = \text{micro}, \\ [-2\mathbf{c}_0 N^{2/3+c_2}, 2\mathbf{c}_0 N^{2/3+c_2}] \setminus I_{\text{micro}} & \text{for } * = \text{meso}, \\ (-\infty, N] \setminus (I_{\text{micro}} \cup I_{\text{meso}}) & \text{for } * = \text{macro}. \end{cases} \quad (138)$$

Similar to the proof of Theorem 2.6 in Section 3.1, we show that for any fixed M and $\epsilon > 0$,

$$\left| \mathbb{P} \left(\frac{G_{(N,N)}^{f_N, \text{micro}}(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x \right) - \mathbb{P} \left(\max_{s \in [-M, M]} (\mathcal{A}(s) - s^2 + \ell(s)) \leq x \right) \right| < \epsilon \quad (139)$$

for all N large enough, independent of the particular formula of f_N . Then we show that for any fixed $\epsilon > 0$, for all N large enough, independent of the particular formula of f_N ,

$$\mathbb{P} \left(\frac{G_{(N,N)}^{f_N, \text{meso}}(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} > x \right) < \epsilon, \quad (140)$$

and at last show that for any fixed $\epsilon > 0$, for all N large enough, independent of the particular formula of f_N ,

$$\mathbb{P} \left(\frac{G_{(N,N)}^{f_N, \text{macro}}(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} > x \right) < \epsilon. \quad (141)$$

Thus we prove Theorem 2.11.

To prove (139), we note that by the symmetry of the lattice,

$$\mathbb{P} \left(\frac{G_{(N,N)}^{f_N, \text{micro}}(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x \right) = \mathbb{P} \left(\max_{s \in [-M, M]} (\tilde{H}_N(s) + \ell(s) + l_N(s)) \leq x \right), \quad (142)$$

where $\tilde{H}_N(s)$ is defined in (100), with $\alpha = 2/3$ and $\ell_N(s) = \mathbf{c}_0 s$. Then using the convergence results in Theorem 2.18 and Proposition 2.3, we derive (139) by the argument similar to those in Section 3.1.1.

To prove (140), we use a simple inequality that for any lattice points (x_0, y_0) , (x, y) and (x', y') such that $x_0 \geq x \geq x'$ and $y_0 \geq y \geq y'$, we have

$$G_{(x_0, y_0)}(x, y) \leq G_{(x_0, y_0)}(x', y') - G_{(x, y)}(x', y'). \quad (143)$$

Now we take $(x_0, y_0) = (N, N)$, $(x, y) = (0, s)$ where the integer $s \in I_{\text{meso}}$ and corresponding to (x, y) , with the same s ,

$$(x', y') = \begin{cases} (-[M\mathbf{c}_0 N^{2/3}] - \frac{s}{2}, -[M\mathbf{c}_0 N^{2/3}] + \frac{s}{2}) & \text{if } s \text{ is even,} \\ (-[M\mathbf{c}_0 N^{2/3}] - \frac{s-1}{2}, -[M\mathbf{c}_0 N^{2/3}] + \frac{s+1}{2}) & \text{if } s \text{ is odd.} \end{cases} \quad (144)$$

It is easy to see that if we prove that if M is large enough, then for all large enough N ,

$$\mathbb{P} \left(\max_{s \in \mathbb{Z} \text{ and } s \in I_{\text{meso}}} G_{(N,N)}(x', y') \geq \mathbf{a}_0 N + \mathbf{a}_0 M \mathbf{c}_0 N^{2/3} + x \mathbf{b}_0 N^{1/3} \right) < \frac{\epsilon}{2}, \quad (145)$$

and uniformly for all $s \in \mathbb{Z} \cap I_{\text{meso}}$, if N is large enough,

$$\mathbb{P} \left(G_{(x,y)}(x', y') \leq \mathbf{a}_0 M \mathbf{c}_0 N^{2/3} - f_N \left(\frac{s}{2\mathbf{c}_0 N^{2/3}} \right) \right) < \frac{\epsilon}{2} \frac{1}{4\mathbf{c}_0 N^{2/3+c_2}}, \quad (146)$$

then (140) is proved.

The inequality (145) is analogous to (124) and can be proved by the arguments used in Section 3.1.2. The inequality (146) is a direct consequence of Proposition 2.1(b). Then the proof of Theorem 2.11 is complete.

To prove (141), we estimate the probability that the point-to-point LPP $\mathbb{P}(G_{(N,N)}(0, s) - f(N) > \mathbf{b}_0 N^{1/3} x)$ by Lemma 2.2 for all $s \in \mathbb{Z} \cap I_{\text{macro}}$, and then sum up all these probabilities as an upper bound of the left-hand side of (141). The argument is similar to the proof of (110) and the detail is omitted.

The proof of Theorem 2.13 is similar. We divide the \perp -shaped path \tilde{L} defined in (81) into the ‘‘micro’’, ‘‘meso’’ and ‘‘macro’’ parts according to the distance to the corner $(0, 0)$, and use the three methods to estimate the point-to-curve LPP between (N, N) to them, as in the proof of Theorem 2.11. We omit the detail.

4 Proofs of Corollaries 2.9 and 2.14

4.1 Proof of Corollary 2.14(a) and (b)

Parts (a) and (b) of Corollary 2.14 are direct consequences of Theorems 2.11 and 2.13 respectively. We only give detail of the proof of part (a), since that of part (b) is similar.

Define the random function $f(x)$ on the domain $D = [0, \infty)$ by

$$f(x) = -\check{G}(k-1, x) \quad (147)$$

where the weight on the lattice is assumed to be inhomogeneous and the weights $w(i, j)$ with $i = 1, \dots, k$ are specified by (89). Then the point-to-point LPP $\check{G}(N, N)$ is expressed as

$$\check{G}(N, N) = \max_{x \in [0, N]} G_{(N, N)}(k, x) - f(x). \quad (148)$$

We see that the $\check{G}(N, N)$ on the lattice with inhomogeneous weights has the same distribution as $G_{(N-k, N)}^f(L)$, where the notation is the same as in Theorem 2.11. As $N \rightarrow \infty$, we have (\check{G} is defined in (88))

$$\lim_{N \rightarrow \infty} \mathbb{P}(\check{G}_N \leq x) = \lim_{N \rightarrow \infty} \mathbb{P}\left(\frac{G_{(N, N)}^f(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x\right). \quad (149)$$

Although the random function $f(x)$ is not in the form of $f_N(x)$ in (78), the different is only a constant term. We write for any N

$$f(2s\mathbf{c}_0 N^{\frac{2}{3}}) = -s\mathbf{a}_0 \mathbf{c}_0 N^{\frac{2}{3}} - \ell^{(N)}(s) \mathbf{d}_0 N^{\frac{1}{3}}. \quad (150)$$

For any $\epsilon > 0$, by choosing the constant C properly, the inequality

$$\ell^{(N)}(s) < C + \frac{1}{2}s^2 \quad (151)$$

is satisfied in probability at least $1 - \epsilon$. So by Theorem 2.11, given any $\epsilon > 0$, for large enough N

$$\left| \mathbb{P}\left(\frac{G_{(N, N)}^f(L) - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} \leq x\right) - \mathbb{P}\left(\max_{s \in (0, \infty)} (\mathcal{A}(s) - s^2 + \ell^{(N)}(s)) \leq x\right) \right| < \epsilon. \quad (152)$$

Furthermore, it is not hard to see that the random function $\ell^{(N)}(s)$ converges weakly to

$$\sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) - 4 \sum_{i=1}^k w_i(s_i - s_{i-1}) - s_k^2 \quad (153)$$

on any compact interval. At last, the weak convergence of $\ell^{(N)}(s)$, together with the estimate (151) and Proposition 2.16(b), implies that

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{P}\left(\max_{s \in (0, \infty)} (\mathcal{A}(s) - s^2 + \ell^{(N)}(s)) \leq x\right) = \\ \mathbb{P}\left(\max_{0=s_0 \leq s_1 \leq \dots \leq s_k \leq M} \left(\mathcal{A}(s_k) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) - 4 \sum_{i=1}^k w_i(s_i - s_{i-1}) - s_k^2\right) \leq x\right). \end{aligned} \quad (154)$$

Combining (149), (152) and (154), we prove Corollary 2.14(a).

4.2 Proof of Corollary 2.14(c)

Let the weights $w(i, j)$ be defined as in Corollary 2.14(c). Define the stochastic processes $B_{1,N}, \dots, B_{k,N}$ as

$$B_{i,N}(s) = \begin{cases} \frac{1}{\mathbf{b}_0 N^{1/3}} \left(G_{([\alpha N]+i, [\alpha N])}([\alpha N] + i, \alpha N + 2\mathbf{c}_0 N^{2/3} s) - \mathbf{a}_0 \mathbf{c}_0 N^{2/3} s \right) & \text{if } s \geq 0, \\ \frac{1}{\mathbf{b}_0 N^{1/3}} \left(-\check{G}_{([\alpha N]+i, [\alpha N])}([\alpha N] + i, \alpha N + 2\mathbf{c}_0 N^{2/3} s) - \mathbf{a}_0 \mathbf{c}_0 N^{2/3} s \right) & \text{if } s < 0. \end{cases} \quad (155)$$

Then we have the weak convergence

$$B_{i,N}(s) \Rightarrow \sqrt{2} \mathbf{B}_i(s) + 4w_i s \quad (156)$$

on any compact interval as $N \rightarrow \infty$, where $\mathbf{B}_1(s), \dots, \mathbf{B}_k(s)$ are independent two-sided Brownian motions.

Next define the stochastic processes

$$A_N^{(1)}(s) = \frac{1}{\mathbf{b}_0 N^{1/3}} \left(\check{G}_{([\alpha N], [\alpha N])}([\alpha N], [\alpha N] - 2\mathbf{c}_0 N^{2/3} s) - \mathbf{a}_0 (\alpha N - \mathbf{c}_0 N^{2/3} s) \right), \quad (157)$$

$$A_N^{(2)}(s) = \frac{1}{\mathbf{b}_0 N^{1/3}} \left(G_{(N,N)}([\alpha N] + k + 1, [\alpha N] + 2\mathbf{c}_0 N^{2/3} s) - \mathbf{a}_0 (\alpha N - \mathbf{c}_0 N^{2/3} s) \right), \quad (158)$$

By Theorem 2.18 and Proposition 2.3, we have the weak convergence that on any interval $[-M, M]$ as $N \rightarrow \infty$

$$A_N^{(1)}(s) \Rightarrow \alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s) - \frac{s^2}{\alpha}, \quad A_N^{(2)}(s) \Rightarrow (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \alpha)^{-2/3} s) - \frac{s^2}{1 - \alpha}, \quad (159)$$

where $\mathcal{A}^{(1)}(s)$ and $\mathcal{A}^{(2)}(s)$ are two independent Airy processes.

We denote the three regions of \mathbb{R}^{k+1}

$$\begin{aligned} R_1(M) &= \{(s_0, s_1, \dots, s_k) \mid -M \leq s_0 \leq s_1 \leq \dots \leq s_k \leq M\}, \\ R_2(M) &= \{(s_0, s_1, \dots, s_k) \mid s_0 \leq s_1 \leq \dots \leq s_k \leq M \text{ and } s_0 < -M\}, \\ R_3(M) &= \{(s_0, s_1, \dots, s_k) \mid -M \leq s_0 \leq s_1 \leq \dots \leq s_k \text{ and } s_k > M\}, \end{aligned} \quad (160)$$

and write

$$\frac{\check{G}_{(N,N)} - \mathbf{a}_0 N}{\mathbf{b}_0 N^{1/3}} = \max \left(G_N^{(1)}(M), G_N^{(2)}(M), G_N^{(3)}(M) \right), \quad (161)$$

where for $i = 1, 2, 3$,

$$\begin{aligned} G_N^{(i)}(M) &= \frac{1}{\mathbf{b}_0 N^{1/3}} \max_{(s_0, \dots, s_k) \in R_i(M)} \left(\check{G}_{([\alpha N], [\alpha N])}([\alpha N], [\alpha N] + [2\mathbf{c}_0 N^{2/3} s_0]) \right. \\ &\quad \left. + \sum_{i=1}^k \check{G}_{([\alpha N]+i, [2\mathbf{c}_0 N^{2/3} s_{i-1}])}([\alpha N] + i, [2\mathbf{c}_0 N^{2/3} s_i]) \right. \\ &\quad \left. + G_{(N,N)}([\alpha N] + k + 1, [2\mathbf{c}_0 N^{2/3} s_k]) - \mathbf{a}_0 N \right) \\ &= \max_{(s_0, \dots, s_k) \in R_i(M)} \left(A_N^{(1)} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_0]}{2\mathbf{c}_0 N^{2/3}} \right) + A_N^{(2)} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_k]}{2\mathbf{c}_0 N^{2/3}} \right) \right. \\ &\quad \left. + \sum_{i=1}^k \left(\tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_i]}{2\mathbf{c}_0 N^{2/3}} \right) - \tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_{i-1}] - 1}{2\mathbf{c}_0 N^{2/3}} \right) \right) \right). \end{aligned} \quad (162)$$

It is a direct consequence of the convergence results (156) and (159) that for any $M > 0$

$$\begin{aligned} \lim_{N \rightarrow \infty} \mathbb{P} \left(G_N^{(1)}(M) \leq x \right) &= \mathbb{P} \left(\max_{-M \leq s_0 \leq s_1 \leq \dots \leq s_k \leq M} \left(\alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s_0) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) \right. \right. \\ &\quad \left. \left. + (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \beta)^{-2/3} s_k) - 4 \sum_{i=1}^k w_i (s_i - s_{i-1}) - \frac{s_0^2}{\alpha} - \frac{s_k^2}{1 - \alpha} \right) \leq x \right). \end{aligned} \quad (163)$$

To estimate $G_N^{(2)}(M)$, we recall the Hypothesis 2.10 for Theorem 2.11 and define the function (cf. (79) with $c_1 = 1/2$ and $c_4 = 1/100$)

$$L_N(x) = N \max \left(\mathbf{a}_0 - \frac{\mathbf{a}_0 x / N}{2} - \frac{1}{2} \mathbf{d}_0 \left(\frac{x / N}{2 \mathbf{c}_0} \right)^2, a_0 \left(1 - \frac{x}{N} \right) + \frac{1}{100} \left| \frac{x}{N} \right| \right), \quad (164)$$

and then the function depending on a constant K to be determined below by (171) (cf. (21) with $C = K$ and $c_1 = 1/2$)

$$\ell_N^{\max}(s) = \max \left(K + \frac{s^2}{2}, \frac{1}{\mathbf{d}_0 N^{1/3}} \left(\mathbf{a}_0 N - s \mathbf{a}_0 \mathbf{c}_0 N^{2/3} - L_N(2s \mathbf{c}_0 N^{2/3}) \right) \right). \quad (165)$$

Then we have $\mathbb{P} \left(G_N^{(2)}(M) \geq x \right) \leq \mathbb{P}_1 + \mathbb{P}_2 + \mathbb{P}_3 + \mathbb{P}_4 + \mathbb{P}_5$, where

$$\mathbb{P}_1 = \mathbb{P} \left(\max_{s_0 < -M} A_N^{(1)} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_0]}{2\mathbf{c}_0 N^{2/3}} \right) \geq -\alpha^{1/3} \ell_N^{\max}(\alpha^{-2/3} s_0) \right), \quad (166)$$

$$\mathbb{P}_2 = \mathbb{P} \left(\max_{s_k \in \mathbb{R} \setminus (-M, M)} A_N^{(2)} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_k]}{2\mathbf{c}_0 N^{2/3}} \right) \geq -(1 - \alpha)^{1/3} \ell_N^{\max}((1 - \alpha)^{-2/3} s_k) \right), \quad (167)$$

$$\mathbb{P}_3 = \mathbb{P} \left(\max_{s_k \in [-M, M]} A_N^{(2)} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_k]}{2\mathbf{c}_0 N^{2/3}} \right) \geq K \right), \quad (168)$$

$$\mathbb{P}_4 = \mathbb{P} \left(\max_{\substack{s_0 \leq \dots \leq s_k \\ s_0 < -M, \\ s_k \in [-M, M]}} \sum_{i=1}^k \left(\tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_i]}{2\mathbf{c}_0 N^{2/3}} \right) - \tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_{i-1}] - 1}{2\mathbf{c}_0 N^{2/3}} \right) \right) \geq \alpha^{1/3} \ell_N^{\max}(\alpha^{-2/3} s_0) - K + x \right), \quad (169)$$

$$\begin{aligned} \mathbb{P}_5 &= \mathbb{P} \left(\max_{\substack{s_0 \leq \dots \leq s_k \\ s_0 < -M, \\ |s_k| \in [M, (1 - \alpha)N / (2\mathbf{c}_0 N^{2/3})]}} \sum_{i=1}^k \left(\tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_i]}{2\mathbf{c}_0 N^{2/3}} \right) - \tilde{B}_{i,N} \left(\frac{[2\mathbf{c}_0 N^{2/3} s_{i-1}] - 1}{2\mathbf{c}_0 N^{2/3}} \right) \right) \right. \\ &\quad \left. \geq \alpha^{1/3} \ell_N^{\max}(\alpha^{-2/3} s_0) + (1 - \alpha)^{1/3} \ell_N^{\max}((1 - \alpha)^{-2/3} s_k) + x \right). \end{aligned} \quad (170)$$

Now we assume $\epsilon > 0$ is a small constant. By the property of the Airy process in Lemma 2.16 and the convergence (158), we have that there exists an $K > 0$ depending on ϵ such that for all interval $[-M, M]$, the inequality

$$\mathbb{P}_3 < \epsilon. \quad (171)$$

As in the proof of Theorem 2.11, we have that if M is large enough, then for all large enough N ,

$$\mathbb{P}_1 < \epsilon, \quad \mathbb{P}_2 < \epsilon. \quad (172)$$

Also By standard argument for random walk, we find that if K depends on ϵ as in (171) but not M , and M is large enough, then for all N large enough,

$$\mathbb{P}_4 < \epsilon, \quad \text{and} \quad \mathbb{P}_5 < \epsilon. \quad (173)$$

Hence we conclude that if M is large enough, then for all N large enough,

$$\mathbb{P}\left(G_N^{(2)}(M) \geq x\right) < 5\epsilon. \quad (174)$$

By a parallel argument, we have that if M is large enough, then for all N large enough,

$$\mathbb{P}\left(G_N^{(3)}(M) \geq x\right) < 5\epsilon. \quad (175)$$

Finally, by (163), (174) and (175), together with the result that is a consequence of Proposition (2.16)(b),

$$\begin{aligned} & \lim_{M \rightarrow \infty} \mathbb{P} \left(\max_{-M \leq s_0 \leq s_1 \leq \dots \leq s_k \leq M} \left(\alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s_0) + \sqrt{2} \sum_{i=1}^k \mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1}) \right. \right. \\ & \quad \left. \left. + (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \beta)^{-2/3} s_k) - 4 \sum_{i=1}^k w_i(s_i - s_{i-1}) - \frac{s_0^2}{\alpha} - \frac{s_k^2}{1 - \alpha} \right) \leq x \right) \\ &= \mathbb{P} \left(\max_{-\infty \leq s_0 \leq s_1 \leq \dots \leq s_k \leq \infty} \left(\alpha^{1/3} \mathcal{A}^{(1)}(\alpha^{-2/3} s_0) + \sqrt{2} \sum_{i=1}^k (\mathbf{B}_i(s_i) - \mathbf{B}_i(s_{i-1})) \right. \right. \\ & \quad \left. \left. + (1 - \alpha)^{1/3} \mathcal{A}^{(2)}((1 - \beta)^{-2/3} s_k) - 4 \sum_{i=1}^k w_i(s_i - s_{i-1}) - \frac{s_0^2}{\alpha} - \frac{s_k^2}{1 - \alpha} \right) \leq x \right), \end{aligned} \quad (176)$$

we prove part (c) of Corollary 2.14.

4.3 Proof of Corollary 2.9

Since all the five parts of the corollary are similar, we only prove part (e) and the proofs to the other four parts are analogous or easier.

The random function $h^{\text{flat/Bern}}$ in (58) defines a random polygonal chain $L^{\text{flat/Bern}}$ by (35), we define a function $\ell^{(N)}(s)$ associated to it by the relation

$$L^{\text{flat/Bern}} = \{(s \mathbf{c}_0 N^{2/3} - \ell^{(N)}(s) \mathbf{d}_0^* N^{1/3}, -s \mathbf{c}_0 N^{2/3} - \ell^{(N)}(s) \mathbf{d}_0^* N^{1/3}) \mid s \in \mathbb{R}\}. \quad (177)$$

Then $\ell^{(N)}(s)$ is a continuous function such that it is deterministic for $s < 0$ and random for $s > 0$. It is clear that for $s > 0$, $\ell^{(N)}(s)$ is mapped to the path of a simple symmetric random walk, such that

$$\left(2\mathbf{d}_0^* N^{1/3} \ell^{(N)} \left(\frac{k}{2\mathbf{c}_0 N^{2/3}} \right), k = 1, 2, \dots \right) \sim \left(\sum_{i=1}^k X_i, k = 1, 2, \dots \right), \quad (178)$$

where X_i are in i.i.d. distribution with $\mathbb{P}(X_i = -1) = \mathbb{P}(X_i = 1) = 1/2$.

Now we let $c_1 = 1/2$, $c_2 = 1/6$, $c_3^* = 1/100$. We have that for any $\epsilon > 0$, there is a large enough constant C_ϵ such that if we let $C = C_\epsilon$ and c_1, c_2, c_3, c_4 defined above, then in probability greater than $1 - \epsilon$, inequality (21) is satisfied by $\ell^{(N)}(s)$ on $[-N^{c_2}, N^{c_2}]$ and conditions (49) and (50) for L^* are also satisfied by $L^{\text{flat/Bern}}$. To check it, we note that the part of L where the x -coordinate is negative does not violate inequality (21), (49) and (50), while on the other part of L , we simply use the property of simple symmetric random walk.

By Theorem 2.8, we have that if the coefficients C, c_1, c_2, c_3^* are chosen as above, then for large enough N

$$\left| \mathbb{P} \left(\frac{h^{\text{flat/Bern}}(2\sigma\mathbf{c}_0 N^{2/3}; \mathbf{a}_0^* N) - 2N}{2\mathbf{d}_0^* N^{1/3}} > -x \right) - \mathbb{P} \left(\max_{s \in \mathbb{R}} \mathcal{A}(s) - (s - \sigma)^2 + \ell^{(N)}(s) < x \right) \right| < \epsilon, \quad (179)$$

where $\mathcal{A}(s)$ is an Airy process.

It is clear that as $N \rightarrow \infty$, on $(-\infty, 0]$, $\ell^{(N)}(s)$ uniformly converges to the constant function 0. On the other hand, for positive s , by the correspondence (178) and Donsker's theorem, $\ell^{(N)}(s)$ weakly converges to $\sqrt{2}q^{-1/4}\mathbf{B}(s)$, where $\mathbf{B}(s)$ is a standard Brownian motion and the constant factor is the ratio

$$\sqrt{2}q^{-1/4} = \frac{\sqrt{2\mathbf{c}_0 N^{2/3}}}{2\mathbf{d}_0^* N^{1/3}} \quad (180)$$

by (17) and (48). By argument like that between (152) and (154) in the proof of Corollary 2.14(a), we prove part (e) of Corollary 2.9.

5 Proof of Theorem 2.18

Let $C'_N = [(C + 1)N^\alpha]/N^\alpha$ which depends on N and lies in the interval $[C, C + 1]$. Define

$$H_{N,\pm}(s) := \frac{1}{\mathbf{b}_0 N^{1/3}} \left(\check{G} \left(N \pm 2C'_N N^\alpha + s\mathbf{c}_0 N^{2/3}, N \pm 2C'_N N^\alpha - s\mathbf{c}_0 N^{2/3} \right) - \mathbf{a}_0 \left(N \pm 2C'_N N^\alpha \right) \right) \quad (181)$$

for all $s \in [-M, M]$. It is a direct to check that

$$H_{N,\pm}(s) = (1 + 2C'_N N^{\alpha-1})^{-1/3} H_{N \pm 2C'_N N^\alpha} (s + O(N^{\alpha-1})), \quad (182)$$

where the term $O(N^{\alpha-1})$ is independent of s .

We first prove the following claim.

Claim 5.1. *For any given $\epsilon, \delta > 0$, there exists a constant N_1 which only depends on M, α and C such that*

$$\mathbb{P} \left(\max_{s \in [-M, M]} |H_{N,\pm}(s) - H_N(s)| \geq \frac{\delta}{2} \right) < \frac{\epsilon}{2} \quad (183)$$

for all $N > N_1$.

To see this we first note that $H_N(s)$ is tight (see [26, Lemma 5.3.]), i.e., there exist constant $\delta' > 0$ and $N'_1 > 0$ which only depend on M, ϵ and δ such that

$$\mathbb{P} \left(\max_{|s_1|, |s_2| \leq M, |s_1 - s_2| \leq \delta'} |H_N(s_1) - H_N(s_2)| \geq \frac{\delta}{6} \right) < \frac{\epsilon}{6} \quad (184)$$

for all $N \geq N'_1$.

The relation (182) implies that $H_{N, \pm}(s)$ are also tight. Therefore there exist constant $\delta'' > 0$ and $N''_1 > 0$ which only depend on M, ϵ and δ such that

$$\mathbb{P} \left(\max_{|s_1|, |s_2| \leq M, |s_1 - s_2| \leq \delta''} |H_{N, \pm}(s_1) - H_{N, \pm}(s_2)| \geq \frac{\delta}{6} \right) < \frac{\epsilon}{6} \quad (185)$$

for all $N \geq N''_1$.

Now we fix δ' and δ'' , and denote $t_j = j \cdot \min\{\delta', \delta''\}$ for all integers j such that $-M \leq t_j \leq M$. By the slow decorrelation of LPP (see, [15, Theorem 2.1]), we know that there exists some constant N'''_1 which depends on $C, \epsilon, \delta, \delta'$ and δ'' such that

$$\mathbb{P} \left(\max_{-M \leq |j| \cdot \min\{\delta', \delta''\} \leq M} |H_{N, \pm}(t_j) - H_N(t_j)| \geq \frac{\delta}{6} \right) < \frac{\epsilon}{6} \quad (186)$$

for all $N \geq N'''_1$.

Note that for all $s \in [-M, M]$, there exists some j such that $|t_j - s| \leq \min\{\delta', \delta''\}$, and that

$$|H_{N, \pm}(s) - H_N(s)| \leq |H_{N, \pm}(t_j) - H_N(t_j)| + |H_{N, \pm}(s) - H_{N, \pm}(t_j)| + |H_N(s) - H_N(t_j)|. \quad (187)$$

Together with (184), (185) and (186) we obtain Claim 5.1.

Now we prove Theorem 2.18. Note that for $s \in [-M, M]$ such that $s\mathbf{c}_0 N^{2/3} \in \mathbb{Z}$ we have

$$\begin{aligned} & \check{G} \left(N + 2C'_N N^\alpha + s\mathbf{c}_0 N^{2/3}, N + 2C'_N N^\alpha - s\mathbf{c}_0 N^{2/3} \right) \\ & - \check{G} \left(N + l_N(s) N^\alpha + s\mathbf{c}_0 N^{2/3}, N + l_N(s) N^\alpha - s\mathbf{c}_0 N^{2/3} \right) \\ & = G_{(N+2C'_N N^\alpha + s\mathbf{c}_0 N^{2/3}, N+2C'_N N^\alpha - s\mathbf{c}_0 N^{2/3})} \left(N + l_N(s) N^\alpha + s\mathbf{c}_0 N^{2/3}, N + l_N(s) N^\alpha - s\mathbf{c}_0 N^{2/3} \right) \end{aligned} \quad (188)$$

which has the same distribution as $\check{G}((2C'_N - l_N(s)) N^\alpha, (2C'_N - l_N(s)) N^\alpha)$. If $\alpha > 1/3$, by applying Proposition 2.1(b) we obtain the following estimate

$$\mathbb{P} \left(H_{N, +}(s) - \tilde{H}_N(s) \leq -\frac{\delta}{2} \right) \leq e^{-c' N^{1-\alpha}} \quad (189)$$

for all $N \geq N'_2$, where c' and N'_2 are positive parameters independent of s . If $\alpha \leq 1/3$, we have

$$\mathbb{P} \left(H_{N, +}(s) - \tilde{H}_N(s) \leq -\frac{\delta}{2} \right) \leq \mathbb{P} \left(H_{N, +}(s) - \tilde{H}_N(s) \leq -\frac{\delta}{2} N^{(3\alpha-2)/6} \right). \quad (190)$$

By applying Proposition 2.1(b) again, we obtain

$$\mathbb{P} \left(H_{N, +}(s) - \tilde{H}_N(s) \leq -\frac{\delta}{2} N^{(3\alpha-2)/6} \right) \leq e^{-c'' N^{\alpha/2}}, \quad (191)$$

for all $N \geq N_2''$, where c'' and N_2'' are positive parameters independent of s . Therefore we still have the estimate (190) with c' and N_2' replaced by c'' and N_2'' . By combining the above two cases we have

$$\mathbb{P} \left(\max_{s \in [-M, M], \mathbf{sc}_0 N^{2/3} \in \mathbb{Z}} \left(\tilde{H}_N(s) - H_{N,+}(s) \right) \geq \frac{\delta}{2} \right) \leq \sum_{s \in [-M, M], \mathbf{sc}_0 N^{2/3} \in \mathbb{Z}} e^{-c'' N^{\min\{1-\alpha, \alpha/2\}}} \quad (192)$$

for all $N \geq N_2''' = \max\{N_2', N_2''\}$, where $c''' = \min\{c', c''\}$. Note that the above estimate includes all the lattice points on the path $\{(N + \mathbf{sc}_0 N^{2/3}, N - \mathbf{sc}_0 N^{2/3}) \mid s \in [-M, M]\}$. Similarly one can obtain an analogous estimate including all the lattice points on the path $\{(l_N(s)N^\alpha + \mathbf{sc}_0 N^{2/3}, l_N(s)N^\alpha - \mathbf{sc}_0 N^{2/3}) \mid s \in [-M, M]\}$. Moreover, the right hand side of the estimate tends to zero as $N \rightarrow \infty$ since there are only $o(N)$ terms in the summation. As a result, there exists an integer N_2 which depends on M, C, ϵ , and δ such that

$$\mathbb{P} \left(\max_{\substack{s \in [-M, M], \\ \text{lattice points}}} \left(\tilde{H}_N(s) - H_{N,+}(s) \right) \geq \frac{\delta}{2} \right) < \frac{\epsilon}{2} \quad (193)$$

for all $N \geq N_2$, where the maximum is taken over all the $s \in [-M, M]$ such that $(\mathbf{sc}_0 N^{2/3}, -\mathbf{sc}_0 N^{2/3})$ or $(l_N(s)N^\alpha + \mathbf{sc}_0 N^{2/3}, l_N(s)N^\alpha - \mathbf{sc}_0 N^{2/3})$ is a lattice point. One can remove this restriction by using the definition of H_N and $H_{N,+}$, and replacing the value of \tilde{G} at an arbitrary point by the interpolation of that on two nearby lattice points. Therefore there exists an integer N_2 which depends on M, C, ϵ such that

$$\mathbb{P} \left(\max_{s \in [-M, M]} \left(\tilde{H}_N(s) - H_{N,+}(s) \right) \geq \frac{\delta}{2} \right) < \frac{\epsilon}{2} \quad (194)$$

for all $N \geq N_2$.

By combing this estimate and Claim 5.1, we immediately have

$$\begin{aligned} & \mathbb{P} \left(\max_{s \in [-M, M]} \left(\tilde{H}_N(s) - H_N(s) \right) \geq \delta \right) \\ & \leq \mathbb{P} \left(\max_{s \in [-M, M]} \left(\tilde{H}_N(s) - H_{N,+}(s) \right) \geq \frac{\delta}{2} \right) + \mathbb{P} \left(\max_{s \in [-M, M]} \left(H_{N,+}(s) - H_N(s) \right) \geq \frac{\delta}{2} \right) \\ & < \epsilon \end{aligned} \quad (195)$$

for all $N \geq \max\{N_1, N_2\}$.

Similarly, there exists an integer N_3 which depends on M, C, ϵ and δ such that

$$\mathbb{P} \left(\max_{s \in [-M, M]} \left(H_{N,-}(s) - \tilde{H}_N(s) \right) > \frac{\delta}{2} \right) < \frac{\epsilon}{2} \quad (196)$$

for all $N \geq N_3$. By combing this estimate and Claim 5.1, we have

$$\begin{aligned} & \mathbb{P} \left(\max_{s \in [-M, M]} \left(H_N(s) - \tilde{H}_N(s) \right) \geq \delta \right) \\ & \leq \mathbb{P} \left(\max_{s \in [-M, M]} \left(H_N(s) - H_{N,-}(s) \right) \geq \frac{\delta}{2} \right) + \mathbb{P} \left(\max_{s \in [-M, M]} \left(H_{N,-}(s) - \tilde{H}_N(s) \right) \geq \frac{\delta}{2} \right) \\ & < \epsilon \end{aligned} \quad (197)$$

for all $N \geq \max\{N_1, N_2\}$. Theorem 2.18 follows immediately by taking $N_0 = \max\{N_1, N_2, N_3\}$.

6 Gibbs property of multi-layer discrete PNG and proof of Lemma 2.19

The goal of this section is to prove Lemma 2.19. The proof relies on the correspondence between the LPP model and the multi-layer discrete polynuclear growth (PNG) model. The essential ingredient of the proof is the Gibbs property of the multi-layer discrete PNG model, analogous to the Gibbs property of the nonintersecting Brownian motions studied in [17]. We describe the multi-layer discrete PNG model and its relation to LPP, following closely to the presentation in [26], to facilitate the proof. Then we prove Lemma 2.19 based on technical results in Lemmas 6.2 and 6.3. The strategy of our proof is similar to that of [17, Lemma 5.1].

The multi-layer PNG model can be expressed in terms of the trajectory of particles on \mathbb{Z} conditioned to be non-colliding. The dynamics of a single particle follows a particular form of random walk, such that the particle jumps at integer time t , in the way that in even time t , it can only jump in the positive direction, and at odd time t , it can only jump in the negative direction. At any integer time t , the movement of the particle is

$$\mathbb{P}(\text{the particle jumps } k \text{ steps}) = (1 - \sqrt{q})(\sqrt{q})^k, \quad k = 0, 1, 2, \dots, \quad (198)$$

and the steps are upward or downward depending on the parity of t . We call a particle following this law a *PNG particle*, and represent the trajectory of a PNG particle by an integer-valued, piecewise-constant, right-continuous function whose only discontinuous points are integers. To describe the trajectory of a PNG particle mathematically, we introduce the notation of *PNG trajectory line*. Let I be an interval and $h(t)$ a function defined on I that satisfies

$$h(t) = h(\lceil t \rceil) \in \mathbb{Z}, \quad \text{and} \quad h(2m) \geq h(2m - \epsilon), \quad h(2m + 1) \leq h(2m + 1 - \epsilon), \quad \text{for } m \in \mathbb{Z} \cap I, \quad (199)$$

then we say that $h(t)$ is a PNG trajectory line on I . Thus the trajectory of a PNG particle is uniquely represented by a PNG trajectory line function on the time span.

If two PNG trajectory lines $h(t)$ and $g(t)$ on the same interval I satisfy

$$\limsup_{t \rightarrow t_0} g(t_0) < \liminf_{t \rightarrow t_0} h(t_0) \quad \text{for all } t_0 \in I, \quad (200)$$

we say that $h(t)$ is *above* $g(t)$ and $g(t)$ is *below* $h(t)$.

The multi-layer discrete PNG model is defined by an ensemble of infinitely many PNG particles under the condition that they do not collide with each other and they end at fixed positions after a certain time. Below we give a precise description of the model, with a parameter $N \in \mathbb{Z}_+$ that is related to the total time that the particles move. We fix a constant $\epsilon \in (0, 1)$ throughout this section. We label the infinitely many particles by x_0, x_1, x_2, \dots , and represent them by the PNG trajectory lines $x_0(t), x_1(t), \dots$ on the interval between $-2N + 1 - \epsilon$, the initial time, and $2N - 1 + \epsilon$, the terminal time. We require the initial and terminal conditions

$$x_i(-2N + 1 - \epsilon) = -i, \quad x_i(2N - 1 + \epsilon) = -i, \quad i = 0, 1, 2, \dots \quad (201)$$

We also require the strong non-colliding condition that the PNG trajectory line $x_i(t)$ is above $x_{i+1}(t)$, for all $i = 0, 1, 2, \dots$. The multi-layer discrete PNG model is defined by the trajectories of the particles, one particle for one layer. One example is given in Figure 5. If infinitely many PNG trajectory lines $(x_0(t), x_1(t), \dots)$ on the interval $I = [-2N + 1 - \epsilon, 2N - 1 + \epsilon]$ satisfy (201) and that

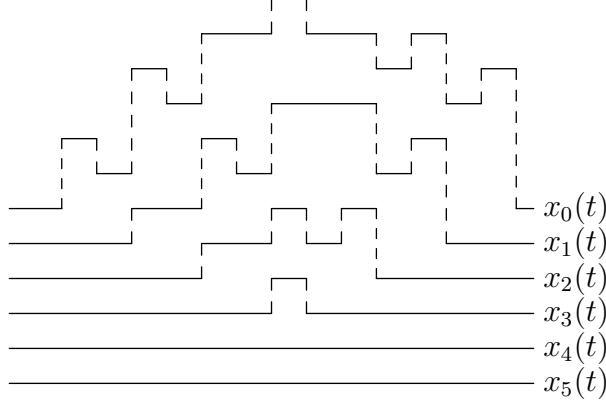


Figure 5: An example of multi-layer discrete PNG with $N = 4$.

$x_i(t)$ is above $x_{i+1}(t)$, we say that these PNG trajectory lines form an N -permissible configuration with parameter N . Thus the multi-layer discrete PNG model is equivalent to the set of permissible configurations. Later we are going to define a probability on it.

Note that in the multi-layer discrete PNG model with parameter N , since the infinitely many particles are densely packed at the initial time and terminal time, the particles x_i , $i = 1, \dots, N-1$, are stationary in time $[-2N+1-\epsilon, -2N+2+2i] \cup (2N-1-2i, 2N-1+\epsilon]$, and the particles x_N, x_{N+1}, \dots are stationary during all the time $[-2N+1-\epsilon, 2N-1+\epsilon]$. Therefore only the first N particles have nontrivial dynamics.

We define a weight w for a PNG trajectory line h on the interval $I = [a, b]$ as

$$w(h) = \prod_{k=[a]+1}^{[b]} p(|h(k) - h(k-\epsilon)|) \quad \text{where} \quad p(k) = \sqrt{1-q}(\sqrt{q})^k, \quad (202)$$

and define the weight for an N -permissible configuration of PNG trajectory lines (h_0, h_1, \dots) (note that $h_n(t) \equiv 0$ for all $n \geq N$)

$$w(h_0, h_1, \dots) = \prod_{k=0}^{N-1} w(h_k), \quad (203)$$

where $I = [-2N+1-\epsilon, 2N-1+\epsilon]$ in the formula of $w(h_i)$. It is clear that the weight defined in (202) of PNG trajectory lines corresponds to the law of the PNG particle, and the weight defined in (203) of the N -permissible configurations corresponds to the law of the multi-layer discrete PNG model. The normalization $\sqrt{1-q}$ is chosen such that (see [26, Claim 3.10, Page 300])

$$\sum_{\text{all } N\text{-permissible configurations}} w(h_0, h_1, \dots) = 1. \quad (204)$$

So the weight (203) defines a probability on the set of all N -permissible configurations.

Thus for any N , the joint distribution of $\check{G}(N+k, N-k)$ for $k = -N, -N+1, \dots, N$, as defined in (4) is the same as the joint distribution of $h_0(2k)$ for $k = -N, -N+1, \dots, N$, if $(h_0(t), h_1(t), \dots)$ is a random N -permissible configuration with probability given in (203). Then the point-to-curve

LPP in Lemma 2.19 is expressed as [26, Proposition 3.11]

$$\begin{aligned} \max_{K_1 \leq s \leq K_2 - c(K_2 - K_1)} G(N + l^0(s) + s, N + l^0(s) - s) &= \max_{K_1 \leq k \leq K_2 - c(K_2 - K_1)} G(N + k, N - k) \\ &\stackrel{d}{=} \max_{K_1 \leq k \leq K_2 - c(K_2 - K_1)} h_0(2k) = \max_{t \in [K_1, K_2 - c(K_2 - K_1)]} h_0(t). \end{aligned} \quad (205)$$

The proof of Lemma 2.19 relies on the Gibbs property of the probability space of permissible $2N$ -tuples, in particular the Gibbs property as follows.

Lemma 6.1. *Consider $t_1 < t_2$, with $t_1, t_2 \in (-2N + 1 - \epsilon, 2N - 1 + \epsilon)$ and Consider $\tilde{h}(\cdot) = (\tilde{h}_0(\cdot), \tilde{h}_1(\cdot), \dots)$ distributed according to the multi-layer discrete PNG model. Then the law of \tilde{h}_0 restricted to the interval $[t_1, t_2]$ is distributed according to the PNG trajectory of a single line $h(\cdot)$ on the interval $[t_1, t_2]$ conditioned on $h(t_1) = \tilde{h}_0(t_1)$, $h(t_2) = \tilde{h}_0(t_2)$, and $h(\cdot) > \tilde{h}_1(\cdot)$ on the entire interval.*

Proof. This lemma is a direct consequence of the formulas (202) and (203) that define the probability distribution of PNG trajectory lines and N -permissible configurations. \square

We need two more lemmas. The first is a monotone coupling result:

Lemma 6.2. *Let $t_1 < t_2 < t_3 \in \mathbb{R}$, $a_1, a_2, a_3 \in \mathbb{Z}$ and $\tilde{h}(t)$ be a fixed PNG trajectory line on $[t_1, t_3]$ such that $\tilde{h}(t_1) < a_1$, $\tilde{h}(t_3) < a_3$. Suppose $h(t)$ is a random variable in the space of PNG trajectory lines $H := \{h(t) \text{ on } [t_1, t_3] \mid h(t_1) = a_1, h(t_3) = a_3, \text{ and } h(t) \text{ is above } \tilde{h}(t)\}$ where the probability is given by the weight $w(h)$ as in (202) up to a normalization constant, and suppose $g(t)$ is a random variable in the space of PNG trajectory lines $G := \{g(t) \text{ on } [t_1, t_3] \mid g(t_1) = a_1, g(t_3) = a_3\}$ where the probability is also given by the weight $w(g)$ as in (202) up to a normalization constant. Then it follows that*

$$\mathbb{P}(h(t_2) \geq a_2) \geq \mathbb{P}(g(t_2) \geq a_2). \quad (206)$$

Sketch of proof. In the proof of [17, Lemma 2.6], the result of this lemma is shown to hold if the PNG trajectory line is replaced by the trajectory of a standard random walk. The same method, namely the coupling of Monte-Carlo Markov chains, works in our situation.

We consider a continuous-time Markov chain dynamic on the countable sets H and G . Without loss of generality, we assume that t_1 and t_3 are even integers. To distinguish the time variable of the Markov chain dynamic and the variables of $h(t)$ and $g(t)$, we denote the Markov time as τ , and write the random PNG trajectory lines as $h_\tau(t)$ and $g_\tau(t)$ respectively. The time 0 configuration of $h_0(t)$ is chosen arbitrarily in H and we let $g_0(t) = h_0(t)$. The dynamics of the Markov chain are as follows. For each integer $t_0 \in \{t_1 + 1, t_1 + 2, \dots, t_3 - 1\}$, there is an independent exponential clock which rings at rate 1. For each $\tau > 0$, let $r(\tau)$ be i.i.d. random variables in geometric distribution such that $\mathbb{P}(r(\tau) = k) = (1 - q)q^k$ for $k = 0, 1, 2, \dots$. When the clock labeled by t_0 rings, the random PNG trajectory line $h_\tau(t)$ remains the same for $t \notin [t_0, t_0 + 1)$, and changes the value on $[t_0, t_0 + 1)$ into (1) $\max(h_\tau(t_0 - 1), h_\tau(t_0 + 1) + r(\tau))$ if t_0 is even, or (2) $\min(h_\tau(t_0 - 1), h_\tau(t_0 + 1)) - r(\tau)$ if t_0 is odd. Likewise, according to the same clock, the random PNG trajectory line $g_\tau(t)$ remains the same for $t \notin [t_0, t_0 + 1)$ and changes the value on $[t_0, t_0 + 1)$ into (1) $\max(g_\tau(t_0 - 1), g_\tau(t_0 + 1) + r(\tau))$ if t_0 is even, or (2a) $\min(g_\tau(t_0 - 1), g_\tau(t_0 + 1)) - r(\tau)$ if t_0 is odd and $\min(g_\tau(t_0 - 1), g_\tau(t_0 + 1)) - r(\tau) > \max(\tilde{h}(t_0 - 1), \tilde{h}(t_0 + 1))$, or (2b) remains the same otherwise.

Then we observe that for any $\tau > 0$, $h_\tau(t) \geq g_\tau(t)$ for all $t \in [t_1, t_3]$. Another fact is that the marginal distributions of these time dynamics converge to the invariant measures for this Markov chain, which are given by the weight function (202) on the state spaces G and H respectively. This can be confirmed by checking that the multi-layer PNG model measure is the unique invariant measure under these irreducible, aperiodic Markov dynamics. \square

Lemma 6.3. *Let $t_1 < t_2 < t_3 \in \mathbb{R}$, $a_1, a_3 \in \mathbb{Z}$ and $a_2 \in \mathbb{R}$ such that $(t_1, a_1), (t_2, a_2), (t_3, a_3)$ are collinear, i.e.,*

$$\frac{a_2 - a_1}{t_2 - t_1} = \frac{a_3 - a_2}{t_3 - t_2}. \quad (207)$$

Let $g(t)$ be a random variable in the space of PNG trajectory lines with fixed ends $G := \{g(t) \text{ on } [t_1, t_3] \mid g(t_1) = a_1, g(t_3) = a_3\}$ where the probability is given by the weight $w(g)$ as in (202) up to a normalization constant. Then

$$\mathbb{P}(g(t_2) \geq a_2) \geq \frac{1}{2} - \delta_{\min(t_2-t_1, t_3-t_2)}, \quad (208)$$

where for any $t > 0$, $\delta_t > 0$ is a decreasing function in t and $\delta_t \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Without loss of generality, we assume that $t_1 = a_1 = 0$ and then $a_2 = (t_2/t_3)a_3$. We also assume in the proof that t_1, t_2, t_3 are even integers. Consider the i.i.d. discrete random variables X_1, X_2, \dots with support \mathbb{Z} and distribution

$$\mathbb{P}(X_1 = k) = \frac{1 - \sqrt{q}}{1 + \sqrt{q}} (\sqrt{q})^{|k|}, \quad k = 0, \pm 1, \pm 2, \dots, \quad (209)$$

and define $S_n = \sum_{k=1}^n X_k$. Then the distribution of $g(t_2)$ is the same as the distribution of $S_{t_2/2}$ under the condition that $S_{t_3/2} = a_3$. We take a change of measure, and define another sequence of i.i.d. discrete random variables X'_1, X'_2, \dots with support $\mathbb{Z} - a_3/t_3$ and distribution

$$\mathbb{P}\left(X'_1 = k - \frac{a_3}{t_3}\right) = \frac{(1 - p\sqrt{q})(1 - \sqrt{q}/p)}{1 - q} \times \begin{cases} (p\sqrt{q})^k & \text{if } k \geq 0, \\ (\sqrt{q}/p)^k & \text{if } k < 0, \end{cases} \quad (210)$$

where p is the real number in $(\sqrt{q}, \sqrt{q}^{-1})$ that satisfies

$$\frac{(p^2 - 1)\sqrt{q}}{(1 - p\sqrt{q})(p - \sqrt{q})} = \frac{a_3}{t_3}. \quad (211)$$

Then if we define $S'_n = \sum_{i=1}^n X'_i$, the distribution of $g(t_2) - a_2$ is the distribution of $S'_{t_2/2}$ under the condition that $S'_{t_3/2} = 0$. Explicit computation shows that the mean of X'_1 is zero and the variance of X'_1 is bounded below by a positive constant independent of a_3/t_3 . Thus the random walk with increment X'_k conditioned with $S'_{t_3/2} = 0$ converges weakly to a Brownian motion as $t_3/2 \rightarrow \infty$, and the convergence is uniform in a_3/t_3 . Since for a Brownian bridge from 0 to 0, at any time between the initial and the terminal times, the probability that the position of particle is positive equals 1/2, we have that the probability that $g(t_2) - a_2$ is positive converges to 1/2 as the total steps of the random walk $t_3/2 \rightarrow \infty$ and both $t_2/2 \rightarrow \infty$ and $(t_2 - t_2)/2 \rightarrow \infty$. Since the convergence of the conditioned random walk to a Brownian bridge is uniform in a_3/t_3 , the convergence of $\mathbb{P}(g(t_2) - a_2)$ to 1/2 is also uniform in a_3/t_3 . We thus prove the lemma. \square

Now we can prove Lemma 2.19. By (205), the lemma is transformed into a property of multi-layer discrete PNG model with parameter N . We denote $K'_2 = K_2 - c(K_2 - K_1)$, and let $(h_0(\cdot), h_1(\cdot), \dots)$ be a multi-layer PNG model distributed ensemble of lines with probability defined by (203). Then we have

$$\begin{aligned}
& \mathbb{P} \left(\max_{K_1 \leq k \leq K'_2} h_0(2k) \geq M_1 \right) \\
& \leq \mathbb{P}(h_0(2K_3) < M_3) + \mathbb{P} \left(\max_{K_1 \leq k \leq K'_2} h_0(2k) \geq M_1 \text{ and } h_0(2K_3) \geq M_3 \right) \\
& \leq \mathbb{P}(h_0(2K_3) < M_3) + \sum_{K_1 \leq K \leq K'_2} \sum_{M'_1=M_1}^{\infty} \sum_{M'_3=M_3}^{\infty} \mathbb{P} \left(\begin{array}{c} \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right).
\end{aligned} \tag{212}$$

By Lemmas 6.1 and 6.2, we have the inequality for the conditional probability

$$\mathbb{P} \left(h_0(2K_2) \geq M_2 \left| \begin{array}{c} \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right. \right) \leq \mathbb{P}(g(2K_2) \geq M_2), \tag{213}$$

where $g(t)$ is a random variable in the space of PNG trajectory lines with fixed ends $G := \{g(t) \text{ on } [2K, 2K_3] \mid g(2K) = M'_1, g(2K_3) = M'_3\}$ and the probability is given by the weight $w(g)$ as in (202) up to a normalization constant.

Denote

$$M'_2 = \frac{K_3 - K_2}{K_3 - K} M'_1 + \frac{K_2 - K}{K_3 - K} M'_3, \tag{214}$$

such that $(K, M'_1), (K_2, M'_2), (K_3, M'_3)$ are collinear. It is clear that $M'_2 \geq M_2$, and then by Lemma 6.3

$$\mathbb{P}(g(2K_2) \geq M_2) \geq \mathbb{P}(g(2K_2) \geq M'_2) > \frac{1}{2} - \delta_{\min(K_2-K, K_3-K_2)} > \frac{1}{2} - \delta_{\min(c(K_2-K_1), K_3-K_2)}. \tag{215}$$

where δ_t is the same as in Lemma 6.3.

Thus by (213) and (215),

$$\begin{aligned}
& \mathbb{P} \left(\begin{array}{c} \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right) < \\
& \frac{1}{\frac{1}{2} - \delta_{\min(c(K_2-K_1), K_3-K_2)}} \mathbb{P} \left(\begin{array}{c} h_0(2K_2) \geq M_2, \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right),
\end{aligned} \tag{216}$$

and then

$$\begin{aligned}
& \sum_{K_1 \leq K \leq K'_2} \sum_{M'_1=M_1}^{\infty} \sum_{M'_3=M_3}^{\infty} \mathbb{P} \left(\begin{array}{c} \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right) \\
& < \frac{1}{\frac{1}{2} - \delta_{\min(c(K_2-K_1), K_3-K_2)}} \sum_{K_1 \leq K \leq K'_2} \sum_{M'_1=M_1}^{\infty} \sum_{M'_3=M_3}^{\infty} \mathbb{P} \left(\begin{array}{c} h_0(2K_2) \geq M_2, \max_{K_1 \leq k < K} h_0(2k) < M_1, \\ h_0(2K) = M'_1 \text{ and } h_0(2K_3) = M'_3 \end{array} \right) \\
& \leq \frac{1}{\frac{1}{2} - \delta_{\min(c(K_2-K_1), K_3-K_2)}} \mathbb{P}(h_0(2K_2) \geq M_2).
\end{aligned} \tag{217}$$

Substitute (217) into (212) and use the correspondence (205), we obtain the proof of Lemma 2.19 with the ϵ_t there determined by $2 + \epsilon_t = (\frac{1}{2} - \delta_t)^{-1}$ where δ_t is that in Lemma 6.3.

A Proof of Lemma 2.2

In this appendix we prove the following estimate of $G([\gamma N], N)$:

Lemma A.1. *For any fixed $\gamma_0 > 1$, there exist some constant $L > 0$ and $\delta > 0$ such that*

$$\mathbb{P} \left(G([\gamma N], N) \geq a_0(\gamma)N + sb_0(\gamma)N^{1/3} \right) \leq e^{-cs^{3/2}}, \tag{218}$$

for large N and all $\gamma \in [\gamma_0^{-1}, \gamma_0]$, $s \in [L, \delta N^{2/3}]$. Here $a_0(\gamma)$ and $b_0(\gamma)$ are defined in (6) and (9), $c > 0$ is a constant which only depends on γ_0, L and δ .

Proof. The following formula for the distribution of $G(M, N)$ was known [6]

$$\mathbb{P}(G(M, N) \leq n) = (1 - q)^{MN} D_n(\phi), \tag{219}$$

where $\phi(z) := (1 + \sqrt{q}z)^M (1 + \frac{\sqrt{q}}{z})^N$, and $D_n(\phi)$ is the n -th Toeplitz determinant with symbol ϕ :

$$D_n(\phi) := \det \left(\int_{|z|=1} z^{-j+k} \phi(z) \frac{dz}{2\pi iz} \right)_{j,k=0}^{n-1}. \tag{220}$$

Note that one can take $n \rightarrow \infty$ in (219) and obtain

$$D_\infty(\phi) := \lim_{n \rightarrow \infty} D_n(\phi) = (1 - q)^{-MN}. \tag{221}$$

Now we apply the Geronimo-Case-Borodin-Okounkov formula [23, 12] and obtain

$$\mathbb{P}(G(M, N) \leq n) = D_\infty(\phi)^{-1} D_n(\phi) = \det(1 - K_n), \tag{222}$$

where K_n is an operator on $l^2\{n, n+1, \dots\}$ with kernel

$$K_n(i, j) = \sum_{k=1}^{\infty} U(i, k) V(k, j). \tag{223}$$

Here

$$\begin{aligned} U(i, k) &:= \int_{|z|=1} \left(1 - \frac{\sqrt{q}}{z}\right)^N (1 - \sqrt{q}z)^{-M} z^{-i-k} \frac{dz}{2\pi iz}, \\ V(k, j) &:= \int_{|z|=1} \left(1 - \frac{\sqrt{q}}{z}\right)^{-N} (1 - \sqrt{q}z)^M z^{j+k} \frac{dz}{2\pi iz}. \end{aligned} \quad (224)$$

Now we consider the asymptotics of $\det(1 - K_n)$ when $M = [\gamma N]$, $n = a_0(\gamma)N + sb_0(\gamma)N^{1/3}$ and $N \rightarrow \infty$. Here $\gamma \in [\gamma_0^{-1}, \gamma_0]$ and $s \in [L, \delta N^{2/3}]$ for some parameters $L > 0$ and $\delta > 0$.

Let

$$z_0 := \frac{1 + \sqrt{\gamma q}}{\sqrt{\gamma} + \sqrt{q}}. \quad (225)$$

Note that if we replace the kernels U and V by the following \tilde{U} and \tilde{V} , the determinant $\det(1 - K_n)$ does not change.

$$\begin{aligned} \tilde{U}(i, k) &:= \left(1 - \frac{\sqrt{q}}{z_0}\right)^{-N} (1 - \sqrt{q}z_0)^M z_0^{i+k} U(i, k), \\ \tilde{V}(k, j) &:= V(k, j) \left(1 - \frac{\sqrt{q}}{z_0}\right)^N (1 - \sqrt{q}z_0)^{-M} z_0^{-j-k}. \end{aligned} \quad (226)$$

Write $i = a_0(\gamma)N + xb_0(\gamma)N^{1/3}$, $j = a_0(\gamma)N + yb_0(\gamma)N^{1/3}$ and $k = ub_0(\gamma)N^{1/3}$, where $x, y \geq s$, and $u \geq 0$. Then we have

$$\tilde{U}(i, k) = e^{Nf(z_0)} \int_{|z|=1} e^{(-Nf(z) + N^{1/3}\phi(z))} \frac{dz}{2\pi iz} \quad (227)$$

where

$$f(z) = -\log\left(1 - \frac{\sqrt{q}}{z}\right) + \gamma \log(1 - \sqrt{q}z) + a_0(\gamma) \log z, \quad (228)$$

and $\phi(z) = -(x + u)b_0(\gamma) \log(z/z_0)$.

Note that

$$f'(z) = -\frac{\sqrt{q}}{1-q} \cdot \frac{((\sqrt{\gamma} + \sqrt{q})z - (1 + \sqrt{\gamma q}))^2}{z(z - \sqrt{q})(1 - \sqrt{q}z)}. \quad (229)$$

Therefore near z_0 , we have the following expansions

$$f(z) = f(z_0) - \frac{q^{1/2}(\sqrt{q} + \sqrt{\gamma})^5}{3\gamma^{1/2}(1-q)^3(1 + \sqrt{q\gamma})} (z - z_0)^3 + O(|z - z_0|^4), \quad (230)$$

and

$$\phi(z) = -(x + u) \left(\frac{q^{1/6}(\sqrt{q} + \sqrt{\gamma})^{5/3}}{\gamma^{1/6}(1-q)(1 + \sqrt{q\gamma})^{1/3}} (z - z_0) + O(|z - z_0|^2) \right). \quad (231)$$

Note that one can deform the contour such that the contour intersects a small neighborhood of z_0 . Moreover, for all z on the contour but outside the above neighborhood of z_0 , $\Re(f(z) - f(z_0)) \geq c$

and $\Re\phi(z) \leq -c(x+u)$ for some positive constant c . Therefore by changing the variables near z_0 one can obtain

$$\begin{aligned}\tilde{U}(i, k) &= O(e^{-c\epsilon^3 N}) + b_0(\gamma)^{-1} N^{-1/3} \int_{2\epsilon N^{1/3} e^{-i\pi/3}}^{2\epsilon N^{1/3} e^{i\pi/3}} e^{\frac{1}{3}\xi^3 - (x+u)\xi} \frac{d\xi}{2\pi i} (1 + O(N^{-1/3})) \\ &= O(e^{-c\epsilon^3 N}) + b_0(\gamma)^{-1} N^{-1/3} \text{Ai}(x+u)(1 + O(N^{-1/3})),\end{aligned}\quad (232)$$

where c, ϵ are both positive constants which only depend on L (the lower bound of $x+u$). Similarly we have

$$\tilde{V}(k, j) = O(e^{-c\epsilon^3 N}) + b_0(\gamma)^{-1} N^{-1/3} \text{Ai}(y+u)(1 + O(N^{-1/3})). \quad (233)$$

Hence

$$\begin{aligned}b_0(\gamma) N^{1/3} K_n(i, j) &= O(e^{-c\epsilon^3 N}) + O(N^{1/3} e^{-c\epsilon^3 N}) \int_0^\infty \text{Ai}(y+u) du + O(N^{1/3} e^{-c\epsilon^3 N}) \int_0^\infty \text{Ai}(x+u) du \\ &\quad + \int_0^\infty \text{Ai}(x+u) \text{Ai}(y+u) du (1 + O(N^{-1/3}))\end{aligned}\quad (234)$$

Note that $x, y \geq s \geq L$. By using the asymptotics of the Airy function, we immediately obtain

$$|b_0(\gamma) N^{1/3} K_n(i, j)| \leq e^{-c'(\min\{x^{3/2}, c'N\} + \min\{y^{3/2}, c'N\})} \quad (235)$$

for large enough N, L , where $c', c'' > 0$ are both independent of x, y, γ .

Therefore $|\text{Tr}(K_n^l)| \leq e^{-c'ls^{3/2}}$, $l = 1, 2, \dots$, for large enough N, L , and $s \in [L, \delta N^{2/3}]$, provided $\delta^{3/2} \leq c''$. This estimate implies the following

$$\left| \sum_{k_i \in \{n, n+1, \dots\}, i=1, \dots, l} \frac{1}{l!} \det(K_n(k_i, k_j))_{i,j=1}^l \right| \leq e^{-c'ls^{3/2}}. \quad (236)$$

Hence

$$\mathbb{P}\left(G([\gamma N], N) \geq a_0(\gamma)N + sb_0(\gamma)N^{1/3}\right) = 1 - \det(1 - K_n) \leq \sum_{l=1}^\infty e^{-c'ls^{3/2}} \quad (237)$$

and the lemma follows. \square

References

- [1] Jinho Baik. Painlevé formulas of the limiting distributions for nonnull complex sample covariance matrices. *Duke Math. J.*, 133(2):205–235, 2006.
- [2] Jinho Baik, Gérard Ben Arous, and Sandrine Péché. Phase transition of the largest eigenvalue for nonnull complex sample covariance matrices. *Ann. Probab.*, 33(5):1643–1697, 2005.
- [3] Jinho Baik, Percy Deift, Ken T.-R. McLaughlin, Peter Miller, and Xin Zhou. Optimal tail estimates for directed last passage site percolation with geometric random variables. *Adv. Theor. Math. Phys.*, 5(6):1207–1250, 2001.

- [4] Jinho Baik, Patrik L. Ferrari, and Sandrine Péché. Limit process of stationary TASEP near the characteristic line. *Comm. Pure Appl. Math.*, 63(8):1017–1070, 2010.
- [5] Jinho Baik and Eric M. Rains. Limiting distributions for a polynuclear growth model with external sources. *J. Statist. Phys.*, 100(3-4):523–541, 2000.
- [6] Jinho Baik and Eric M. Rains. Algebraic aspects of increasing subsequences. *Duke Math. J.*, 109(1):1–65, 2001.
- [7] Gérard Ben Arous and Ivan Corwin. Current fluctuations for TASEP: a proof of the Prähofer-Spohn conjecture. *Ann. Probab.*, 39(1):104–138, 2011.
- [8] Alexei Borodin, Patrik L. Ferrari, Michael Prähofer, and Tomohiro Sasamoto. Fluctuation properties of the TASEP with periodic initial configuration. *J. Stat. Phys.*, 129(5-6):1055–1080, 2007.
- [9] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. Large time asymptotics of growth models on space-like paths. II. PNG and parallel TASEP. *Comm. Math. Phys.*, 283(2):417–449, 2008.
- [10] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. Transition between Airy_1 and Airy_2 processes and TASEP fluctuations. *Comm. Pure Appl. Math.*, 61(11):1603–1629, 2008.
- [11] Alexei Borodin, Patrik L. Ferrari, and Tomohiro Sasamoto. Two speed TASEP. *J. Stat. Phys.*, 137(5-6):936–977, 2009.
- [12] Alexei Borodin and Andrei Okounkov. A Fredholm determinant formula for Toeplitz determinants. *Integral Equations Operator Theory*, 37(4):386–396, 2000.
- [13] Ivan Corwin. The Kardar-Parisi-Zhang equation and universality class. *Random Matrices Theory Appl.*, 1(1):1130001, 76, 2012.
- [14] Ivan Corwin, Patrik L. Ferrari, and Sandrine Péché. Limit processes for TASEP with shocks and rarefaction fans. *J. Stat. Phys.*, 140(2):232–267, 2010.
- [15] Ivan Corwin, Patrik L. Ferrari, and Sandrine Péché. Universality of slow decorrelation in KPZ growth. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(1):134–150, 2012.
- [16] Ivan Corwin and Alan Hammond. KPZ line ensemble, 2013. arXiv:1312.2600.
- [17] Ivan Corwin and Alan Hammond. Brownian Gibbs property for Airy line ensembles. *Invent. Math.*, 195(2):441–508, 2014.
- [18] Ivan Corwin and Jeremy Quastel. Renormalization fixed point of the kpz universality class, 2011. arXiv:1103.3422.
- [19] Ivan Corwin, Jeremy Quastel, and Daniel Remenik. Continuum statistics of the Airy_2 process. *Comm. Math. Phys.*, 317(2):347–362, 2013.
- [20] Patrick L. Ferrari. Slow decorrelations in Kardar–Parisi–Zhang growth. *Journal of Statistical Mechanics: Theory and Experiment*, 2008(07):P07022, 2008.

- [21] Patrik L. Ferrari and Herbert Spohn. A determinantal formula for the GOE Tracy-Widom distribution. *J. Phys. A*, 38(33):L557–L561, 2005.
- [22] Patrik L. Ferrari and Herbert Spohn. Scaling limit for the space-time covariance of the stationary totally asymmetric simple exclusion process. *Comm. Math. Phys.*, 265(1):1–44, 2006.
- [23] J. S. Geronimo and K. M. Case. Scattering theory and polynomials orthogonal on the unit circle. *J. Math. Phys.*, 20(2):299–310, 1979.
- [24] T. Imamura and T. Sasamoto. Fluctuations of the one-dimensional polynuclear growth model with external sources. *Nuclear Phys. B*, 699(3):503–544, 2004.
- [25] Kurt Johansson. Shape fluctuations and random matrices. *Comm. Math. Phys.*, 209(2):437–476, 2000.
- [26] Kurt Johansson. Discrete polynuclear growth and determinantal processes. *Comm. Math. Phys.*, 242(1-2):277–329, 2003.
- [27] T. Kriecherbauer and J. Krug. A pedestrian’s view on interacting particle systems, kpz universality, and random matrices. *J. Phys. A*, 43:403001, 2010.
- [28] J. Krug, P. Meakin, and T. Halpin-Healy. Amplitude universality for driven interfaces and directed polymers in random media. *Phys. Rev. A*, 45:638–653, 1992.
- [29] Michael Prähofer and Herbert Spohn. Current fluctuations for the totally asymmetric simple exclusion process. In *In and out of equilibrium (Mambucaba, 2000)*, volume 51 of *Progr. Probab.*, pages 185–204. Birkhäuser Boston, Boston, MA, 2002.
- [30] Michael Prähofer and Herbert Spohn. Scale invariance of the PNG droplet and the Airy process. *J. Statist. Phys.*, 108(5-6):1071–1106, 2002. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.
- [31] Jeremy Quastel. Introduction to KPZ. In *Current developments in mathematics, 2011*, pages 125–194. Int. Press, Somerville, MA, 2012.
- [32] Jeremy Quastel and Daniel Remenik. Supremum of the Airy_2 process minus a parabola on a half line. *J. Stat. Phys.*, 150(3):442–456, 2013.
- [33] Jeremy Quastel and Daniel Remenik. Airy processes and variational problems. In Alejandro F. Ramírez, Gérard Ben Arous, Pablo A. Ferrari, Charles M. Newman, Vladas Sidoravicius, and Maria Eulália Vares, editors, *Topics in Percolative and Disordered Systems*, volume 69 of *Springer Proceedings in Mathematics & Statistics*, pages 121–171. Springer New York, 2014.
- [34] T. Sasamoto. Spatial correlations of the 1D KPZ surface on a flat substrate. *J. Phys. A*, 38(33):L549–L556, 2005.
- [35] Herbert Spohn. KPZ scaling theory and the semi-discrete directed polymer model, 2012. arXiv:1201.0645.
- [36] Craig A. Tracy and Harold Widom. On orthogonal and symplectic matrix ensembles. *Comm. Math. Phys.*, 177(3):727–754, 1996.